The Twin Paradox: A Detailed Study

Andrew A. Gray
Avtec Corporation, 818 Brentwood, Austin, TX 78757
e-mail: andrew@avtecaustin.com

A detailed explanation of the Twin Paradox found in special relativity.
Keywords: Twin Paradox, Special Relativity, Acceleration Required to Turn Around

1. Introduction

Wikipedia describes the twin paradox like this [1]:

"The twin paradox is a thought experiment in special relativity, in which a twin makes a journey into space in a high-speed rocket and returns home to find he has aged less than his identical twin who stayed on Earth. This result appears puzzling because each twin sees the other twin as traveling, and so, according to an application of time dilation, each should paradoxically find the other to have aged more slowly."

2. The Trip in Doc Arthur's Coordinate System

2.1. The Setup

So let's consider two twin doctors, Doctor Arthur and Doctor Beal, to see what happens in the real world. For this thought experiment, Doc Arthur will stay put and Doc Beal will travel in a rocket out to Planet Zog and return back again. For simplicity, we will let Doc Beal start with a large constant velocity, $\beta = 0.885$ (we let $c = 1$). When Doc Beal passes Doc Arthur, they will synchronize their clocks to both read $t = 0$. Then Doc Beal will travel to Planet Zog, turn around with a very quick constant acceleration (we let $a = 1$), then return home, also with the same constant velocity in the opposite direction. Let us first consider this trip in the inertial coordinates of Doc Arthur. We let these coordinates be the "unbarred coordinates" $\{t, x\}$. (Doc Beal's coordinates will be the barred coordinates: **B**arred, **B**eal, get it? We will consider these coordinates later).

We let both Doc Arthur and Doc Beal sit at the origins of their coordinate systems. So for both doctors, their proper time will be the same as their time coordinate:

$$\tau_A = t_A , \quad (1)$$

$$\tau_B = \bar{t}_B . \quad (2)$$

For Doc Arthur, we have the following coordinates at the start:

$$t_{A_o} = \bar{t}_{A_o} = 0 , \quad (3)$$

$$x_{A_o} = \bar{x}_{A_o} = 0 . \quad (4)$$

For Doc Beal, we have the following coordinates at the start:

$$t_{B_o} = \bar{t}_{B_o} = 0 , \quad (5)$$

$$x_{B_o} = \bar{x}_{B_o} = 0 . \quad (6)$$

2.2. The Outgoing Constant Velocity Section

Next, we want to investigate Doc Beal's proper time along his outgoing trip. At the start of the trip, in Doc Arthur's "unbarred coordinates" we have

$$t_{B_o} = 0 , \quad (7)$$

$$x_{B_o} = 0 . \quad (8)$$

And along the outgoing constant velocity route, Doc Beal has simply:

$$t_{B_o} = \frac{1}{\beta} x_B \quad (\text{"going"}), \quad (9)$$

where we have let $\beta = 0.885$. Now we can transform this equation to get Doc Beal's coordinates along his constant velocity route, using the general Lorentz transformation. Transforming, we get Doc Beal's world line towards Zog ("going"):

$$\bar{t}_B - \bar{t}_{B_o} = \gamma\left(t_B - t_{B_o}\right) - \gamma\beta\left(x_B - x_{B_o}\right) \quad (\text{"going"}), \quad (10)$$

$$\bar{x}_B - \bar{x}_{B_o} = \gamma\left(x_B - x_{B_o}\right) - \gamma\beta\left(t_B - t_{B_o}\right) \quad (\text{"going"}). \quad (11)$$

Remembering that both Docs synchronized their clocks to 0 at the origin of both coordinates gives: $t_{B_o} = \bar{t}_{B_o} = 0$, and $x_{B_o} = \bar{x}_{B_o} = 0$. Then plugging in we get:

$$\bar{t}_B - \bar{t}_{B_o} = \gamma\left(t_B\right) - \gamma\beta\left(x_B\right) \quad (\text{"going"}), \quad (12)$$

$$\bar{x}_B - \bar{x}_{B_o} = \gamma\left(x_B\right) - \gamma\beta\left(t_B\right) \quad (\text{"going"}), \quad (13)$$

where $\beta = 0.885$ and $\gamma = 2.148$.

We are only interested in $\bar{t}_B$, so we finally get

$$\bar{t}_B - \bar{t}_{B_o} = 2.148 t_B - 1.90 x_B . \quad (14)$$

So we now can plot Doc Beal's proper time along the outgoing part of the trip, see Fig. 1. (This is just conventional special relativity. We have done nothing out of the ordinary here). For each event $\{t_B, x_B\}$, we have a corresponding $\bar{t}_B = \tau_B$. If we substitute for x_B from equation (9) into equation (14) we get that:

$$\bar{t}_B = \frac{t_B}{\gamma} . \quad (15)$$

Fig. 1 shows these proper times, $\bar{t}_B$, along Doc Beal's outgoing world line. We see from the diagram that when Doc Arthur's proper time reads $t_A = 48$, Doc Beal's proper time coordinate that corresponds to this time is $\bar{t}_B = 22.35$. Thus, in Doc Arthur's coordinate system, Doc Beal's clock appears to run slow. Doc Arthur has 48 units of proper time passage, while Doc Beal's corresponding time passage is given by 22.35. Thus, it appears that Doc Beal's clock is slow by a factor of γ in Doc Arthur's coordinate system. We should point out that only Doc Arthur thinks that these two events (that are simultaneous in his coordinate system) are actually simultaneous, and when we plot the trip in Doc Beal's coordinates, it will be Doc Arthur's clock that appears to be running slowly instead.

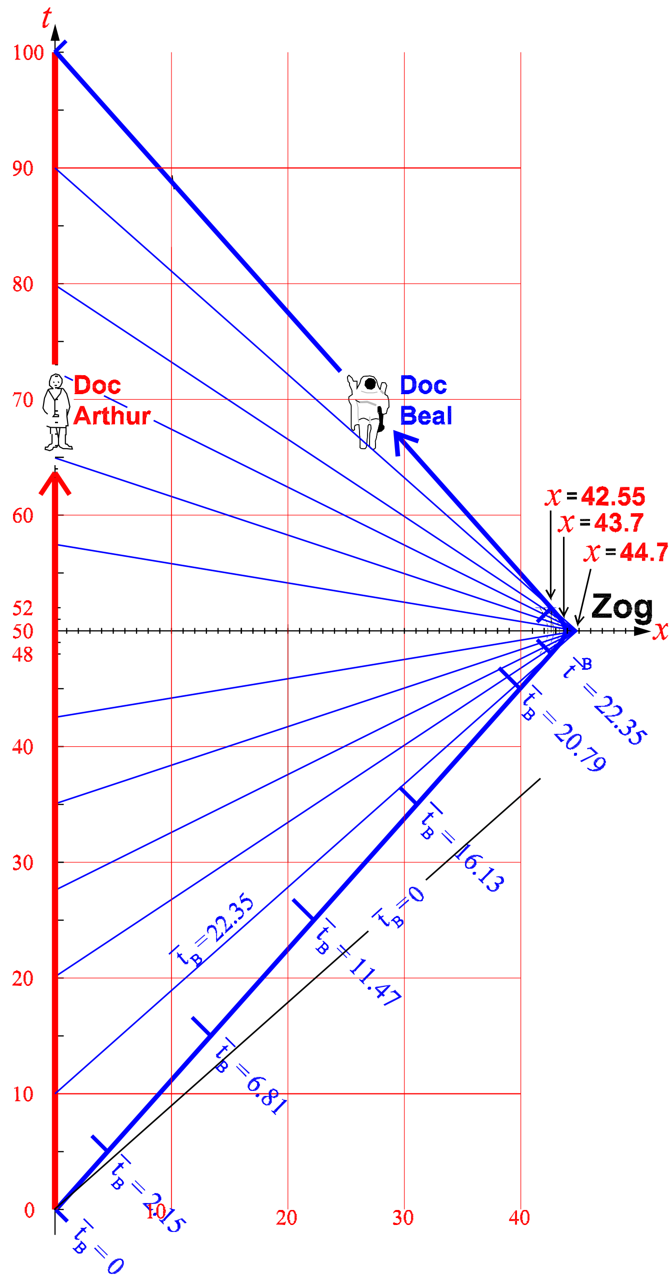

Fig. 1. Spacetime diagram for our twin doctors thought experiment. Doc Beal travels to Zog with velocity $\beta = 0.885$. The trip is shown in Doc Arthur's inertial coordinates $\{t, x\}$. On the outgoing leg of the trip, Doc Beal's proper time $\bar{t}_B$ is shown at regular intervals along the world line. Doc Beal's initial proper time is $\bar{t}_B = 0$, and his start-acceleration proper time (just before Zog) is $\bar{t}_B = 22.35$. So, $\Delta\bar{t}_B = 22.35$. It appears that Doc Beal's clock is running slow compared to Doc Arthur's by a factor of γ. We should note that Doc Arthur thinks that different events are simultaneous from what Doc Beal thinks. We will plot the trip in Doc Beal's coordinates later in the paper, and we will see how it appears that Doc Arthur's clock appears to be running slow by a factor of γ instead. See [2] for a color version of this plot.

2.3. The Short Acceleration Section

We now must consider the short constant acceleration section of Doc Beal's world line. Constant acceleration is easily handled in special relativity. Consider the flat metric:

$$-d\tau^2 = dt^2 + dx^2 + dy^2 + dz^2 \,. \tag{16}$$

Divide this equation by $d\tau^2$, and we get

$$-1 = \frac{dt^2}{d\tau^2} + \frac{dx^2}{d\tau^2} + \frac{dy^2}{d\tau^2} + \frac{dz^2}{d\tau^2} \,. \tag{17}$$

If we note that $\mathbf{u} = \left\{ \frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right\}$, then Eq. (17) can be written:

$$\mathbf{u} \bullet \mathbf{u} = -1 \,, \tag{18}$$

where we have used the Minkowski dot product. If we differentiate equation (18) with respect to τ, we get

$$\mathbf{a} \bullet \mathbf{u} + \mathbf{u} \bullet \mathbf{a} = 0 \,, \tag{19}$$

Or just:

$$\mathbf{u} \bullet \mathbf{a} = 0 \,. \tag{20}$$

Now we can find the magnitude of the acceleration if we note that for an observer at the origin of his coordinate system, we have that $u^{\bar{x}} = u^{\bar{y}} = u^{\bar{z}} = 0$. So if $\mathbf{u} \bullet \mathbf{a} \equiv 0$, then this implies that we must have $a^{\bar{t}} = 0$ in a Lorentz frame instantaneously co-moving with the accelerated observer. So in this co-moving Lorentz frame, the components of the acceleration are given by $\left\{0, a^{\bar{x}}, a^{\bar{y}}, a^{\bar{z}}\right\}$, and the magnitude of the acceleration in this frame can be computed by the simple invariant

$$\mathbf{a}^2 = \mathbf{a} \bullet \mathbf{a} = a^{\bar{x}} a^{\bar{x}} + a^{\bar{y}} a^{\bar{y}} + a^{\bar{z}} a^{\bar{z}} \tag{21}$$

So now let us consider this observer, Doc Beal, who feels an acceleration of constant magnitude, in the x direction. Then the invariant magnitude squared of the acceleration will then just be $\mathbf{a}^2 = (a^{\bar{x}})^2$. We also let $u^{\bar{y}} = u^{\bar{z}} = \bar{y} = \bar{z} = 0$.

So now we can write down a set of linear equations for the constant acceleration motion of Doc Beal, *but now in Doc Arthur's coordinate system:*

$$\mathbf{u} \bullet \mathbf{u} = u^t u^t + u^x u^x = -1 \,, \tag{22}$$

$$\mathbf{u} \bullet \mathbf{a} = u^t a^t + u^x a^x = 0 \,, \tag{23}$$

$$\mathbf{a} \bullet \mathbf{a} = -a^t a^t + a^x a^x = a^2 \,. \tag{24}$$

Notice that a^t is not necessarily 0 in equation (24) because now we are in the "unbarred coordinates". These are simple linear equations. Solving for the acceleration, we find

$$a^t = a u^x \,, \tag{25}$$

$$a^x = a u^t \,. \tag{26}$$

These are simple differential equations in the form:

$$\frac{du^t}{d\tau} = a u^x \,, \tag{27}$$

$$\frac{du^x}{d\tau} = a u^t \,, \tag{28}$$

and one solution is

$$t = a^{-1}\sinh(a\tau)\,, \tag{29}$$

$$x = a^{-1}\cosh(a\tau)\,, \tag{30}$$

where we let the observer be at the origin of the coordinate system and the initial time be $t = \tau = 0$. A general coordinate system and solution for this observer turns out to be [3]:

$$t = \left(a^{-1} + \bar{x}\right)\sinh\left(a\bar{t}\right), \tag{31}$$

$$x = \left(a^{-1} + \bar{x}\right)\cosh\left(a\bar{t}\right), \tag{32}$$

where equations (31) and (32) become equations (29) and (30) at the origin of the system with $\bar{x} = 0$ and $\bar{t} = \tau$. Another solution exactly midway (where the initial times are not zero) is:

$$t - t_{\text{mid}} = a^{-1}\sinh\left(a\left(\tau - \tau_{\text{mid}}\right)\right), \tag{33}$$

$$x - x_{\text{mid}} = a^{-1}\cosh\left(a\left(\tau - \tau_{\text{mid}}\right)\right), \tag{34}$$

and similarly for the general coordinate system. Now we can get back to Doc Beal. Since Doc Beal's proper time is the same as his coordinate time, we have $\tau_B = t_B$. And remembering that we have let the acceleration be $a = 1$ in the $-x$ direction, (the acceleration is $-a$=-1) we now have:

$$t_B - t_{B_{\text{mid}}} = +\sinh\left(\bar{t}_B - \bar{t}_{B_{\text{mid}}}\right), \tag{35}$$

$$x_B - x_{B_{\text{mid}}} = -\cosh\left(\bar{t}_B - \bar{t}_{B_{\text{mid}}}\right). \tag{36}$$

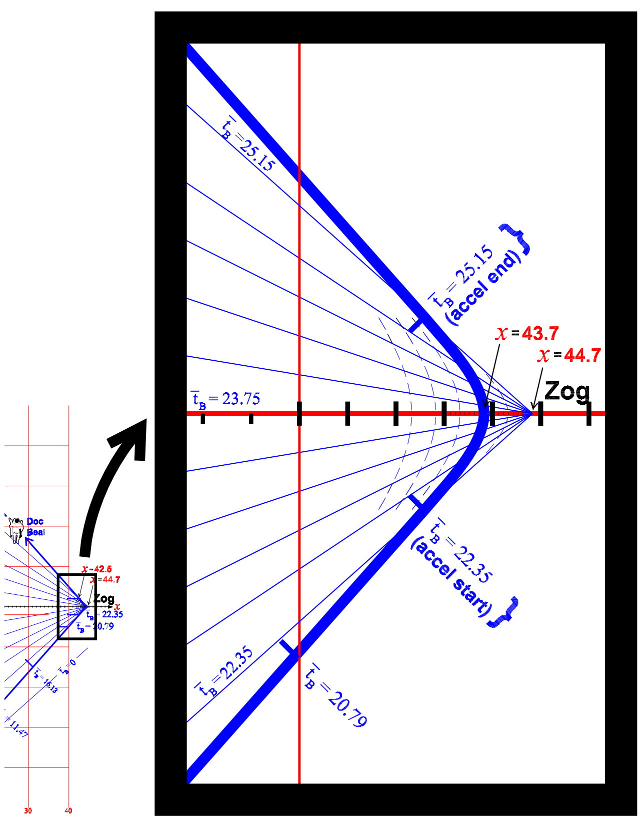

Fig. 2. The acceleration leg of Doc Beal's trip. He starts accelerating at $\bar{t}_B = 22.35$ and finishes at $\bar{t}_B = 25.15$. This gives a net acceleration time of $\Delta\bar{t}_B = 2.8$. (Again, see [2] for a color version of this plot if reading in print).

For this example, $t_{B_{\text{mid}}} = 50$, $x_{B_{\text{mid}}} = 44.7$, and $\bar{t}_{B_{\text{mid}}} = 23.75$ (Figs. 2 and 3). These are Doc Beal's constant acceleration equations:

$$t_B - 50 = +\sinh\left(\bar{t}_B - 23.75\right) \tag{37}$$

$$x_B - 44.7 = -\cosh\left(\bar{t}_B - 23.75\right) \tag{38}$$

A handy formula would be to divide equation (37) by equation (38) and get:

$$\frac{t_B - 50}{x_B - 44.7} = +\tanh\left(\bar{t}_B - 23.75\right) \tag{39}$$

We can now plot these times along the route near Zog (Fig. 2).

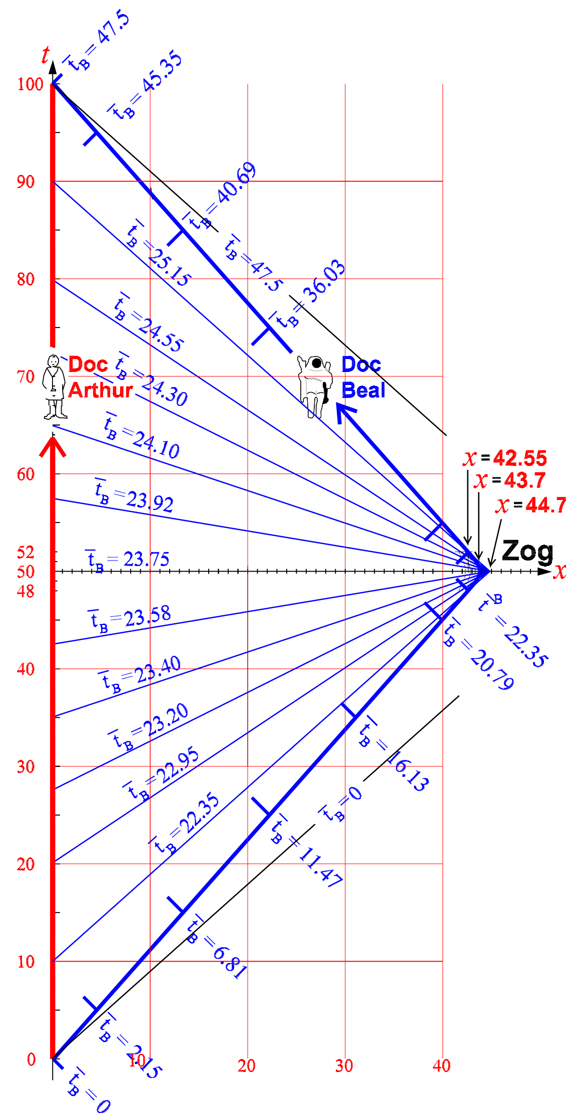

Fig. 3. After obtaining the accelerated portion of the trip, the last leg and the complete trip of Doc Beal can be plotted. Notice that the two constant velocity legs are symmetrical, and have equal proper time durations along their world lines. This calculation shows that Doc Beal's total round trip proper time to be $\Delta\tau_B = 47.5$, and Doc Arthur's total proper time to be $\Delta\tau_B = 100$. Thus, Doc Beal becomes younger than his twin Doc Arthur by a time of 52.5. (Again, see [2] for a color version of this plot if reading in print).

In Fig. 3, we finish the plots and include the proper times along the last leg of Doc Beal's trip. Here is a summary of what we have shown in Figs. 2 and 3:

1. From $\bar{t}_B = 0$ to $\bar{t}_B = 22.35$ Doc Beal uses constant velocity $\beta = 0.885$ towards Zog.
2. From $\bar{t}_B = 22.35$ to $\bar{t}_B = 23.75$ Doc Beal decelerates and stops at Zog.
3. From $\bar{t}_B = 23.75$ to $\bar{t}_B = 25.15$ Doc Beal accelerates back towards Earth until he again is traveling at constant velocity $\beta = 0.885$ towards Earth.
4. By symmetry (around the turning point), the constant velocity leg of the trip back to Earth als takes $\Delta\bar{t}_B = 22.35$, so we can simply add 22.35 to $\bar{t}_B = 25.15$ to get Doc Beal's final arrival proper time at $\bar{t}_{B_{\text{mid}}} = 47.5$. The times in between can similarly be obtained by symmetry (Fig. 3).

This completes the analysis of the trip using Doc Arthur's "unbarred coordinates". Now we want to consider this trip in Doc Beal's "barred coordinates". This will allow us to see how the outgoing constant velocity trips are symmetrical, the acceleration legs are not symmetrical, and actually see what is going on with the simultaneity issues.

3. The Trip in Doc Beal's Coordinate System

3.1. The Setup

In this scenario, we will consider Doc Beal as the one who "stays put", and consider that Doc Arthur moves away to the left and comes back. In this frame, Doc Beal will also see Planet Zog come at him from the right, stop, and go back away from him. For the first "constant velocity" leg of Doc Arthur's trip, the situation is exactly symmetrical. We start with Doc Arthur moving with constant velocity to the left with $\beta = 0.885$. When Doc Arthur passes Doc Beal, they will synchronize their clocks to both read $t = 0$. Then Doc Arthur will travel to the left for a certain amount of time, he will then turn around, then return back to Doc Beal, also with the same constant relative velocity in the opposite direction.

We again let both Doc Arthur and Doc Beal sit at the origins of their coordinate systems. So again for both doctors, their proper time will be the same as their time coordinate:

$$\tau_A = t_A\,, \tag{40}$$

$$\tau_B = \bar{t}_B\,. \tag{41}$$

For Doc Arthur,we have the following coordinates at the start:

$$t_{A_o} = \bar{t}_{A_o} = 0\,, \tag{42}$$

$$x_{A_o} = \bar{x}_{A_o} = 0\,. \tag{43}$$

For Doc Beal, we have the following coordinates at the start:

$$t_{B_o} = \bar{t}_{B_o} = 0\,, \tag{44}$$

$$x_{B_o} = \bar{x}_{B_o} = 0\,. \tag{45}$$

3.2. The Outgoing Constant Velocity Section

Next, we want to investigate Doc Arthur's proper time along his outgoing trip. At the start of the trip, in the "barred coordinates" we have

$$t_{A_o} = 0\,, \tag{46}$$

$$x_{A_o} = 0\,. \tag{47}$$

And along the outgoing constant velocity route, Doc Arthur has simply:

$$t_{A_o} = \frac{1}{\beta} x_A\,, \tag{48}$$

where we have let $\beta = 0.885$. Now we can transform this equation to get Doc Arthur's coordinates along his constant velocity route, using the Lorentz transformation (and skipping the general form this time). Transforming, we get Doc Arthurs's world line away from Doc Beal ("going"):

$$t_A = \gamma\left(\bar{t}_A\right) - \gamma\beta\left(\bar{x}_A\right) \quad (\text{"going"}), \tag{49}$$

$$x_A = \gamma\left(\bar{x}_A\right) - \gamma\beta\left(\bar{t}_A\right) \quad (\text{"going"}) \tag{50}$$

where $\beta = 0.885$ and $\gamma = 2.148$.

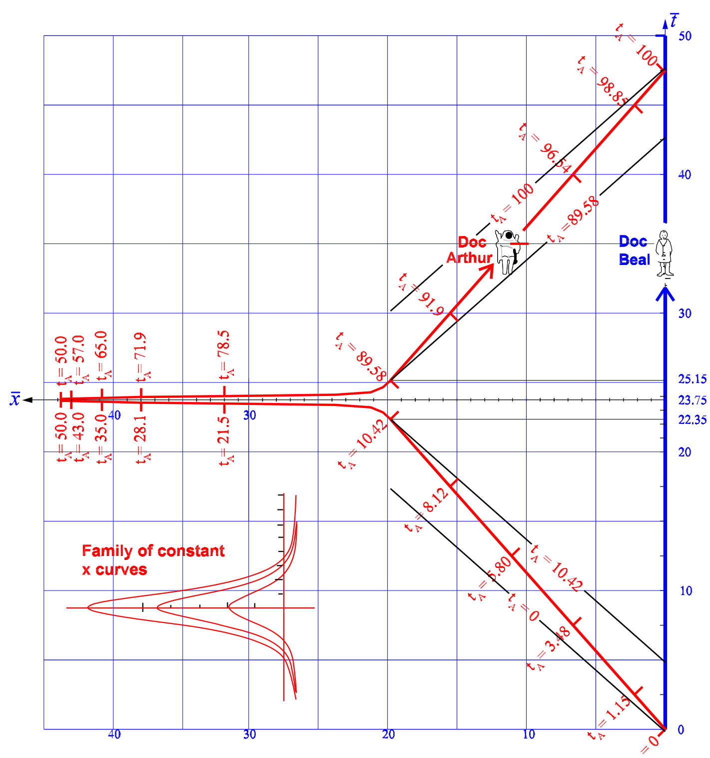

Fig. 4. Doc Arthur's trip plotted in Doc Beal's coordinates. Doc Arthur's proper times, t_A, are plotted along the route. For the constant velocity legs of the world line, Doc Arthur's clock seems to run slow by a factor of γ. The middle section of the world line (the part where Doc Beal uses accelerated coordinates) is a strange shape indeed. In the lower left of the figure, we plot a family of constant x curves so the reader can see the general shape in more detail. The last constant velocity leg of Doc Arthur's trip can again be computed by symmetry. (Again, see [2] for a color version of this plot if reading in print).

We are only interested in t_A, so finally we get:

$$t_A = 2.148\overline{t}_A + 1.90\overline{x}_A \tag{51}$$

Notice that $\overline{x}_A$ will be negative since Doc Arthur is moving to the left. If we substitute for $\overline{x}_A$ from equation (48) into equation (51), we get:

$$t_A = \frac{\overline{t}_A}{\gamma} \tag{52}$$

So we now can plot Doc Arthurs's proper time along the outgoing, constant velocity part of the trip (Fig. 4). Again, this is just conventional special relativity, and we have done nothing out of the ordinary here. For each event $\{\overline{t}_A, \overline{x}_A\}$, we have a corresponding $t_A = \tau_A$. In the lower part of Fig. 4, we can see these proper times t_A, along Doc Arthur's outgoing world line.

We see from the diagram that when Doc Beal's proper time reads $\overline{t}_B = 22.35$, Doc Arthur's proper time coordinate that corresponds to this time is $\overline{t}_A = 10.42$. Thus, in Doc Beals's coordinate system, Doc Arthur's clock appears to run slow. Doc Beal has 22.35 units of proper time passage, while Doc Arthurs's corresponding time passage is given by 10.42. Thus, it appears that Doc Arthurs's clock is slow by a factor of γ in Doc Beal's coordinate system. We should point out that only Doc Beal thinks that these two events (that are simultaneous in his coordinate system) are actually simultaneous, and when we plotted the trip in Doc Arthurs's coordinates, it was Doc Beal's clock that appeared to be running slowly instead.

3.3. Doc Arthur While Doc Beal Feels Acceleration

Now we want to plot Doc Arthur's proper time while Doc Beal feels acceleration. To do this, we need to invert transformations (31) and (32). We get:

$$\overline{t} = \operatorname{arctanh}\left(\frac{t}{x}\right) \tag{53}$$

$$x = -\frac{1}{a} + \sqrt{\overline{x}^2 - \overline{t}^2} \tag{54}$$

and similarly for the transformation with the midpoint not centered on $t = 0$. With this transformation, we can plot the Doc Arthur's proper time along his route while Doc Beal feels acceleration (Fig. 4) Note that Doc Beal stays at the origin of his coordinates. Here is a summary of what we have shown in Fig. 4:

1. From $t_A = 0$ to $t_A = 10.42$ Doc Arthur has constant velocity β = -0.885 away from Doc Beal.
2. From $t_A = 10.42$ to $t_A = 50$ Doc Arthur is in Doc Beal's accelerated coordinate system, and they come to rest relative to one another at $t_A = 50$.
3. From $t_A = 50$ to $t_A = 89.58$ Doc Arthur is still in Doc Beal's accelerated coordinates and he once again ends up traveling with constant velocity back towards Doc Beal.
4. By symmetry (around the midway turning point), the constant velocity leg of the trip back to Doc Beal takes the same amount of time ($\Delta t_A = 10.42$) as the constant velocity leg away from Doc Beal.

3.4. Taking the Limit as the Acceleration Gets Very Large

Now we can see what would happen if we let Doc Beal's acceleration near Planet Zog get very large (so Doc Beal spends little time in the acceleration). From Fig. 3 we see that the limiting trip's first half (to Zog) has $\Delta x_B = 43.7$. So Doc Beal's limiting proper time to Zog, using equation (14), becomes

$$\overline{t}_{B_{Zog}} = \gamma \frac{43.7}{\beta} - \gamma\beta(43.7) = 23.04 \tag{55}$$

Multiplying by 2, Doc Beal's minimum total time for the round trip becomes 46.08.

However, as Doc Beal's proper time during acceleration gets very small, Doc Arthur's proper time during Doc Beal's acceleration is still large, and we cannot ignore this. In the limit, Doc Arthur still spends most of his time in Doc Beal's accelerated coordinate system. Doc Arthur's limiting proper time during Doc Beal's limiting trip to Zog can be computed using equation 9 and simple symmetry:

$$\Delta t = \frac{1}{\beta}\Delta x \qquad \text{(“going”)} \tag{56}$$

So since $\Delta x_{Zog} = 43.7$, and $\beta = 0.885$, we get that Δt_{Zog} = 49.38. Multiplying by 2, we get that Doc Arthur's minimum proper time during the round trip is 98.76.

4. The Triplet "Triodox" (Using Three Doctors)

The Twin Paradox if often expressed in terms of triplets instead of twins, to eliminate the need for acceleration at Planet Zog. We introduce a long lost 3rd sibling of Doc Arthur and Doc Beal named Doc Carter. They are really triplets (Doc Carter was secretly separated from them at birth). The setup for this thought experiment is as follows: The first triplet, Doc Arthur "stays put" at home. The 2nd, Doc Beal takes off for Zog in the same way as the first scenario, with a velocity of $v = \beta c = 0.885c$. Doc Arthur and Doc Beal synchronize their clocks to $t = 0$ as Doc Beal flies by. When Doc Beal gets to Zog, the 3rd triplet, Doc Carter, is already flying by back towards Earth with a velocity of $-0.885c$ (relative to Doc Arthur). When Doc Beal and Doc Carter pass by, they synchronize their clocks to Doc Beal's time. We want to know the total passage of proper time for (Doc Beal + Doc Carter) compared to the total passage of time for Doc Arthur. We briefly go over this scenario in Doc Arthur's coordinate system, then Doc Beal's coordinates, and compare them to the answers we got in the Twin Pair o' Docs scenario.

4.1. Triplet Triodox in Doc Arthur's Coordinates

This case is very similar to the situation that we see in Fig. 1 and Fig. 3. The only difference is that there is no acceleration at Zog, and Doc Beal flies past Doc Carter and they synchronize their clocks to keep track of the total proper time along their combined world line. The scenario plotted in Fig. 5.

From Fig. 5, we see that Doc Arthur simply moves along his x_A=0 world line, so his proper time calculation is trivial, 2 x 49.38 = 98.76. Doc Beal's proper time (to Zog) calculation is found using equation (14) and is 23.04. Doc Carter's proper travel time (from Zog) is also 23.04 by symmetry, yielding a total world line proper time of 46.08. This agrees with the result that we got by taking the limit as Doc Arthur's acceleration time at Zog got very small in section 8.

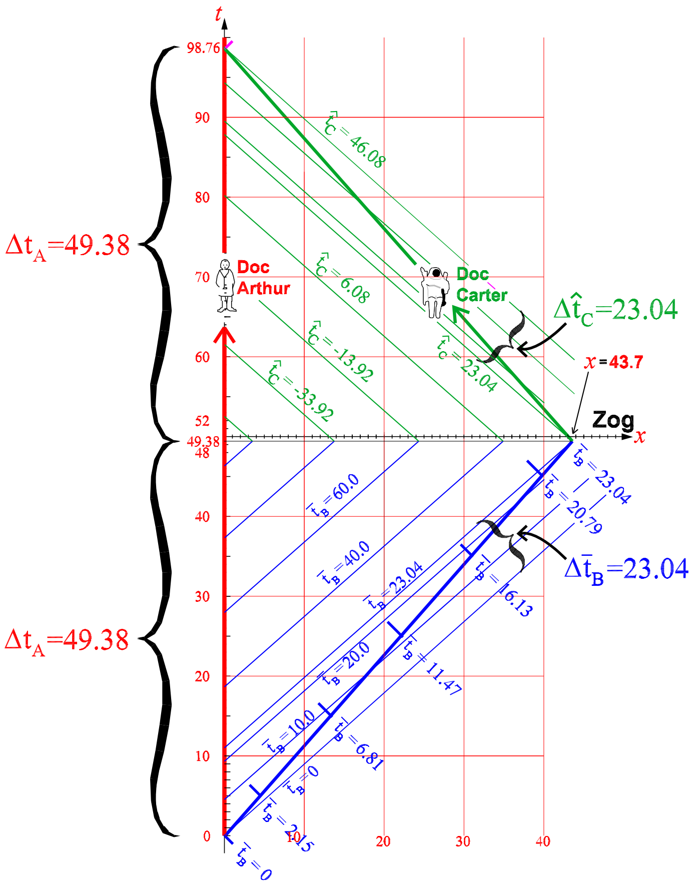

Fig. 5. Our triplet triodox plotted in Doc Arthur's coordinates. We have divided this spacetime diagram around Doc Arthur's midpoint. When Doc Arthur sees Doc Beal arrive at Zog, 49.38 of Doc Arthur's time units have gone by, and 23.04 of Doc Beal's time units have gone by. From the symmetry of the diagram, we see that 23.04 time units of Doc Carter will go by on the return trip to Earth. If we add these up, we again get that the total proper time for Doc Arthur is 98.76 and for the combination of Doc Beal/Doc Carter we have 46.08. Notice that Doc Carter's and Doc Beal's coordinates only match up at time 23.04 in their coordinate systems. At other times the coordinates are discontinuous. Notice that Doc Arthur thinks that Doc Beal's and Doc Carter's clocks run slow by a factor of γ. Also notice that Doc Beal and Doc Carter think that Doc Arthur's clock runs slow by the same factor of γ. (Again, see [2] for a color version of this plot if reading in print).

4.2. Triplet Triodox in Doc Beal's Coordinates

Next, we look at our triplets in terms of Doc Beal's coordinates. In this scenario, Doc Arthur appears to be going "to the left" at 0.885c and Doc Beal is the one who is considered stationary. Then, the Planet Zog appears to come towards Doc Beal from the right. This is when Doc Carter, who appears to be moving to the left even faster than Doc Arthur comes by Doc Beal. When they meet, Doc Carter synchronizes his clock with Doc Beal's clock, and then Doc Carter catches up with Doc Arthur to end the trip.

So first we need to compute Doc Carter's velocity in Doc Beal's frame. We know that Doc Carter is moving to the left at 0.885c in Doc Arthur's frame. And we know that Doc Beal is moving to the right at 0.885c, also in Doc Arthur's frame. So to get Doc Carter's velocity in Doc Beal's frame we use the addition-of-velocities formula found in special relativity. The formula is:

$$\bar{u}_C = \frac{u_C + V_{AB}}{1 + V_{AB} u_C} \tag{57}$$

where $\bar{u}_C$ is Doc Carter's velocity according to Doc Beal, u_C is Doc Carter's velocity according to Doc Arthur, and V_{AB} is the relative velocity between Doc Arthur and Doc Beal. Plugging in what we know gives:

$$\bar{u}_C = \frac{-0.885 - 0.885}{1-(-0.885)(-0.885)} = -0.993 \tag{58}$$

Next we need Doc Arthur's and Doc Carter's equations. We already have worked out Doc Arthur's equations in equations (49), (50), and (51). Doc Carter's equations are similarly:

$$\left(\hat{t}_C - 23.04\right) = \gamma_C\left(\bar{t}_C - 23.04\right) - \gamma_C\beta_C\left(\bar{x}_C\right) \tag{59}$$

$$\hat{x}_C = \gamma_C\left(\bar{x}_C\right) - \gamma_C\beta_C\left(\bar{t}_C - 23.04\right) \tag{60}$$

where the "carrot" (^) on the coordinates refer to Doc Carter's coordinates, and where $\beta c = -0.993c$, which gives $\gamma = 8.47$. Next, similar to equations (15) and (52), we end up with

$$\left(\hat{t}_C - 23.04\right) = \frac{\left(\bar{t}_C - 23.04\right)}{\gamma_C} \tag{61}$$

We can how plot this trip in Fig. 6. Notice that Doc Beal's and Doc Carter's coordinates are not the same, and cannot be combined into a single coordinate system like in Fig. 4.

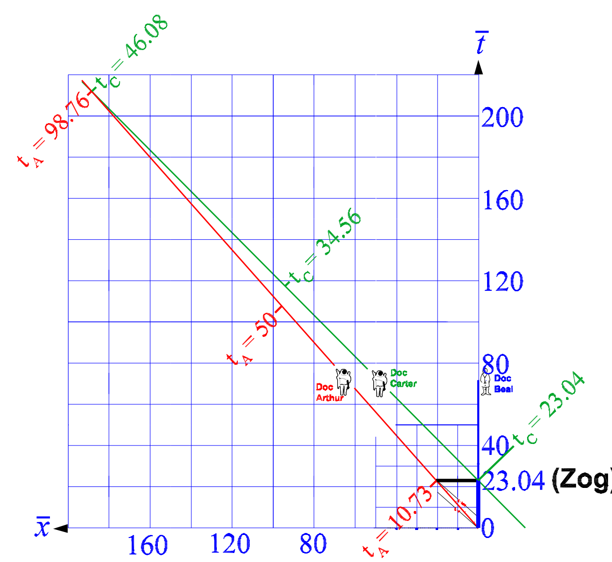

Fig. 6. The triplet trip in Doc Beal's coordinates. Doc Beal stays at the origin of his coordinates. When he gets to Zog at time 23.04, he synchronizes his clock with Doc Carter also at time 23.04. Doc Carter then reaches Doc Arthur at time 46.08 (while Doc Arthur's clock reads 98.76). The computed times agree with all other scenarios shown previously. (Again see [2] for a color version of this plot).

5. Conclusion

In Parts I and II, we see that this twin Pair 'o Docs is not actually a twin paradox. Doc Beal's journey includes acceleration, and is not symmetrical to Doc Arthur's when turned the other way around. Doc Arthur never feels acceleration, and if they both carried an accelerometer with them, Doc Beal would be the only one with a nonzero reading. We see that special relativity handles the situation satisfactorily with acceleration, and even in the limit we see that Doc Beal ends up younger than Doc Arthur with no contradictions.

In Part III, we see that this triplet Tri' o' Docs is not actually a triplet triodox. The Doc Beal/Doc Carter proper time combo duration agrees exactly with the results found in Parts I and II. Doc Arthur's proper time duration agrees as well. We also found that Doc Beal and Doc Carter have discontinuous coordinate systems, so they cannot simply be morphed together, which traditionally has been done to justify the paradox claim.

References

[1] http://en.wikipedia.org/wiki/Twin_paradox.

[2] http://modelofreality.org/TP.pdf.

[3] Charles W. Misner, Kip S. Thorne, John Archibald Wheeler, **Gravitation**, p. 166*ff* (Freeman, San Francisco, 1973).