How do you add relative velocities?*

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Abstract

Following Minkowski [1908], we consider the relative velocity to be the Minkowski space-like vector (and *not* to be the Minkowski bivector as it is in the Hestenes theory [Hestenes 1974]). The Lorentz boost entails the relative velocity (as a space-like Minkowski vector) to be ternary: ternary relative velocity is a velocity of a body with respect to an interior observer *as seen* by a preferred exterior-observer. The Lorentz boosts imply non-associative addition of ternary relative velocities. Within Einstein's special relativity theory, each preferred observer (fixed stars, aether, etc), determine the unique relative velocity among each pair of massive bodies. Therefore, the special relativity founded on Minkowski's axiom, that each pair of reference systems *must* be related by Lorentz isometry, needs a preferred reference system in order to have the unique Einstein's relative velocity among each pair of massive bodies. This choice-dependence of relative velocity violate the Relativity Principle that all reference systems must be equivalent.

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This astonishing conflict of the Lorentz relativity group, with the Relativity Principle, can be resolved in two alternative ways. Either, within Einstein's special relativity theory, to abandon/reject the Relativity Principle in favor of a preferred reference system [Rodrigo de Abreu 2004; Rodrigo de Abreu and Vasco Guerra 2005, 2006]. Or, within the Relativity Principle, replace the Lorentz relativity group by the relativity *groupoid*, with the choice-free binary relative velocities only.

An axiomatic definition of the kinematical unique binary relative velocity as the choice-free the Minkowski space-like vector, leads to the groupoid structure of the set of all deduced relativity transformations (instead of the Lorentz relativity group), with associative addition of binary relative velocities.

Observer-independence, and the Lorentz-invariance, are distinct concepts. This suggest the possibility of formulating many-body relativistic dynamics without Lorentz/Poincare invariance.

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Contents

1 Notation: Minkowski vectors

Given any (non-bound) the Minkowski space-like vector $\mathbf{w}$, one can define a bounded space-like vector, $\mathbf{v}$, as follows,

$$\mathbf{w} \quad \longmapsto \quad \mathbf{v}/c \equiv \mathbf{w}/\sqrt{1+\mathbf{w}^2}, \quad \Longrightarrow \quad \mathbf{v}^2 < c^2. \tag{1.1}$$

The Heaviside-FitzGerald-Lorentz scalar factor is denoted by $\gamma_{\mathbf{v}}$,

$$\gamma_{\mathbf{v}} \equiv \frac{1}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} \quad \Longrightarrow \quad \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1} = \frac{1}{1+\sqrt{1-\frac{\mathbf{v}^2}{c^2}}}$$
$$\Longrightarrow \quad {\gamma_{\mathbf{v}}}^2 - 1 = \mathbf{w}^2 = (\gamma_{\mathbf{v}}\mathbf{v}/c)^2. \tag{1.2}$$

Conversely, given any space-like bounded Minkowski vector $\mathbf{v}$, such that, $\mathbf{v}^2 < c^2$, then, we define unbound vector, denoted by over line, $\overline{\mathbf{v}}$, as follows,

$$\mathbf{v} \quad \longmapsto \quad \overline{\mathbf{v}} \equiv \gamma_{\mathbf{v}}\mathbf{v}/c, \qquad (\overline{\mathbf{v}})^2 = {\gamma_{\mathbf{v}}}^2 - 1, \tag{1.3}$$

$$\mathbf{v}/c \equiv \frac{\overline{\mathbf{v}}}{\sqrt{1+\overline{\mathbf{v}}^2}} \quad \xrightarrow{\text{bijection}} \quad \overline{\mathbf{v}} \equiv \gamma_{\mathbf{v}}\mathbf{v}/c. \tag{1.4}$$

The following identity holds for three arbitrary vectors in arbitrary dimension: for the Grassmann's wedge product, and inner product acting as the graded derivation of the Grassmann algebra,

$$\mathbf{w}\cdot(\mathbf{v}\wedge\mathbf{u})=(\mathbf{w}\cdot\mathbf{v})\,\mathbf{u}-(\mathbf{w}\cdot\mathbf{u})\,\mathbf{v}\qquad(=\mathbf{w}\times(\mathbf{u}\times\mathbf{v})).\tag{1.5}$$

On the right of (1.5) there is the double Gibbs's cross product of vectors that is orientation-dependent. We prefer the orientation-free Grassmann's exterior product, $\mathbf{v}\wedge\mathbf{u}$ is a bi-vector, than, the Gibbs cross internal product, for two reasons. Firstly, in dimension $\neq$ three, Gibbs's product needs the extra explication, like in [Plebański and Przanowski 1988], and, secondly, because of this superfluous orientation-dependence.

In particular (1.5) gives

$$\mathbf{u}\cdot\frac{\mathbf{v}\wedge\mathbf{u}}{c^2}=\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\,\mathbf{u}-\left(1-\frac{1}{\gamma_{\mathbf{u}}^2}\right)\mathbf{v}.\tag{1.6}$$

Sometimes, for simplicity of formulas, the scalar magnitude of the light velocity is set, $\mathbf{c}^2=1$.

2 The Einstein isometric relative velocity needs preferred reference system

The vector space of Grassmann bi-vectors inside of Clifford algebra, is the Lie algebra of the Lie group of isometries. Each bi-vector, $P\wedge Q$, generate an isometry

$$P\wedge Q\quad\hookrightarrow\quad L_{P\wedge Q}\in O(1,3).$$

Following Minkowski in 1908, we identify a reference system with a normalized time-like vector field, $P^2=-1$.

2.1 Definition (The Einstein isometric relative velocity)**.** Let $\{P,A,B\}$ be a three-body massive system given by a set of three time-like vectors, and $\mathbf{v}$ be a bounded space-like, such that $\mathbf{v}\cdot P=0$, and $P^2=-1$. The velocity $\mathbf{v}$ of a Bob B, relative to Alice A, as seen/measured by a preferred observed P, is said to be *isometric*, or *ternary*, or *the Einstein velocity*, if it is defined in terms of the isometric Lorentz boost,

$$L_{P\wedge\overline{\mathbf{v}}}\in O(1,3),\qquad L_{P\wedge\overline{\mathbf{v}}}\,A=B.\tag{2.1}$$

The above definition is motivated by the following theorem.

2.2 Theorem (Isometry-link problem)**.** *For the massive three-body system given in terms of the three time-like vectors* $\{P, A, B\}$*, the Lorentz-boost-link equation for unknown space-like vector* $\mathbf{w}$*,* $L_{P\wedge\mathbf{w}}\, A = B$*, has the unique solution,* $\overline{\mathbf{v}} = \mathbf{w}(P, A, B)$*. This ternary velocity-solution is reciprocal,* $\mathbf{v}(P, A, B) = -\mathbf{v}(P, B, A)$*.*

See Section 5 for outline of a proof. Detailed proof is presented in [Oziewicz 2006].

2.3 Definition (Domain and co-domain)**.** Each space-like vector, $\mathbf{w} \equiv \gamma_{\mathbf{v}}\mathbf{v}/c$, possess the following pair of two-dimensional manifolds of time-like normalized vectors

$$\begin{aligned} D_{\mathbf{w}} &\equiv \{A \in \operatorname{der}\mathcal{F} | A^2 = -1, A \cdot \mathbf{w} = 0\}, \\ C_{\mathbf{w}} &\equiv \{B \in \operatorname{der}\mathcal{F} | B^2 = -1, B \cdot \mathbf{w} = \mathbf{w}^2\}. \end{aligned} \tag{2.2}$$

2.4 Axiom (Binary relative velocity)**.** Let $A \in D_{\mathbf{u}}$ and $\mathbf{u}^2 < c^2$. Then, and *only* then, $\exists !$ body $B = b_{\mathbf{u}}A \equiv \gamma_{\mathbf{u}}(A + \frac{\mathbf{u}}{c}) \in C_{\mathbf{u}}$, moving with a velocity $\mathbf{u}$ *relative* to A. The space-like relative velocity $\mathbf{u}$ is said to be *binary*,

$$\frac{\mathbf{u}}{c} \equiv \frac{\varpi(A, B)}{c} = \frac{B}{-B \cdot A} - A. \tag{2.3}$$

2.5 Corollary. $\gamma_{\mathbf{u}} = -A \cdot B$.

2.6 Theorem. *The Einstein's isometric ternary relative velocity (parameterizing the Lorentz boost) looks like the kind of 'subtraction' of absolute/binary velocities,*

$$\begin{aligned} \mathbf{u}(P, A, B) &= \frac{P \cdot (A + B)\{(P \cdot B)\varpi(P, B) - (P \cdot A)\varpi(P, A)\}}{(P \cdot A)^2 + (P \cdot B)^2 - 1 - A \cdot B} \\ &\simeq i_{gP}\{P \wedge (B - A)\}, \end{aligned} \tag{2.4}$$

$$-B \cdot A = (P \cdot B)(P \cdot A)\left\{1 - \frac{\varpi(P, B) \cdot \varpi(P, A)}{c^2}\right\}, \tag{2.5}$$

$$\begin{aligned} (P \wedge A \wedge B)^2 = {} & (P \cdot A)^2 + (P \cdot B)^2 - 1 \\ & - (P \cdot A)^2 (P \cdot B)^2 \left\{1 - \frac{(\varpi(P, B) \cdot \varpi(P, A))^2}{c^4}\right\}. \end{aligned} \tag{2.6}$$

2.7 Corollary. Consider co-planar system of massive bodies, $P \wedge A \wedge B = 0$. In this particular case the above set of expressions (8.1)-(8.3) is reduced to the expression often presented by Rodrigo de Abreu and Vasco Guerra publications,

$$\mathbf{u}(P, A, B)|_{P \wedge A \wedge B = 0} \; = \; \frac{\varpi(P, B) - \varpi(P, A)}{1 - \varpi(P, B) \cdot \varpi(P, A)/c^2}. \tag{2.7}$$

Minkowski in 1908 identify a reference system with a time-like vector field, and defined the special relativity by means of the following single axiom.

2.8 Axiom (Minkowski axiom)**.** Any two reference systems must be connected by the Lorentz transformation (*i.e.* by the isometry acting on *all* vectors, including not time-like vectors).

The Minkowski axiom gives the cornerstone of XX century physics: Lorentz group-covariance.

The Minkowski axiom does not need explicitly the concept of relative velocity, and leads to choice-dependent, P-dependent, relative velocity among Alice and Bob. Each reference system P (that could be interpreted as the physical fixed stars, aether, etc), gives the unique Einstein's reciprocal relative velocity among each pair of massive bodies,

$$\{A, B\} \quad \xrightarrow{\text{preferred } P} \quad \mathbf{v}(P, A, B). \tag{2.8}$$

Contrary to popular matrix-statement in many textbooks, a Lorentz boost from A to B is not unique (2.1). Lorentz boost is choice-dependent, depends on the choice of the preferred observer P. Equivalently, the Lorentz boost depends on the choice of the non-unique embedding of the rotation group as the sub-group of the Lorentz group, $O(3) \hookrightarrow O(1, 3)$. Each Lorentz boost is given by bi-vector, therefore can not be parameterized by a space-like velocity vector alone. Within Lorentz relativity-group the Minkowski axiom 2.8, when a preferred vector P is *not* chosen, there is a bunch of Lorentz-link transformations from A to B, generated by many different bi-vectors,

$$P \wedge \mathbf{w} \quad \neq \quad P' \wedge \mathbf{w}' \quad \neq \quad \ldots,$$

but there is **no** relative velocities among massive bodies. To have just one Einstein's velocity of Bob relative to Alice, we need to chose some one and only one reference system to be preferred. This choice-dependence of Einstein's relative velocity violate the relativity principle stating that *all* reference systems must be equivalent. Within the Minkowski (Lorentz-relativity-group)-axiom 2.8, this equivalence of all reference systems is not possible: the different choices of a preferred system P in Definition 2.1, leads to distinct Einstein's velocities of Bob relative to Alice.

Theorem 2.2 tells that the Lorentz-boost is *not* unique. It is true that all textbooks of special relativity, starting with Max von Laue ancient text [1911, 1921], and take arbitrary contemporary textbook, for example [Barut 1964, 1980, page 17], define the Lorentz boost as the 'unique' basis-dependent matrix, as follows,

$$\text{Lorentz boost} \quad \equiv \quad \begin{pmatrix} \gamma & -(v/c)\gamma & 0 & 0 \\ -(v/c)\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.9}$$

When accepting a priori the above matrix definition (2.9), it is hard to imagine the existence of the another boost, because the above matrix is fixed by velocity parameter, and there is no option for alternative boost. We like to point that our definition of basis-free P-dependent Lorentz boost (2.1), is reduced to the above 'unique-matrix', in the particular basis, when, $P \simeq (1,0,0,0)$, and $\overline{\mathbf{v}} \simeq (0,-v/c,0,0)$. This explain why someone insists that the 'Lorentz boost is unique!'.

For the given space-like Minkowski vector $\mathbf{w}$, there is the two-dimensional sub-manifold of time-like hyperboloid of possible observers of this vector

$$D_{\mathbf{w}} \equiv \{A^2 = -1, A \cdot \mathbf{w} = 0\}. \tag{2.10}$$

Therefore each observer $P \in D_{\mathbf{w}}$, gives rise to his own isometric Lorentz P-boost $L_{P \wedge \mathbf{w}}$, therefore the Lorentz boost is not unique.

In this note we distinguish conceptually the *relativity*-group, from the concept of a symmetry-group. The Lorentz group is a symmetry-group of the metric of the empty space-time, it is a group of isometries. The Lorentz group is also a symmetry-*sub*group of conformal symmetry of the Maxwell equations. However, a priori, the relativity-group of transformations among massive reference systems need not to be a symmetry-group of some other mathematical structure. We remaind that, for example, the light-like vectors does not represent reference systems, therefore, a priori, they does not need to be in the domain of relativity transformations. The present paper deals exclusively with the Lorentz group as the relativity-group.

In fact, the main problem is *not* about the special relativity identified with the Minkowski axiom 2.8, it is *not* about the interpretation of the Relativity Principle, it is *not* about the clock's synchronization, it is *not* so much about of the time measurements, it is *not* about one-way & two-way light velocity. The main problem is the coordinate-free definition of the concept of the relative **velocity**. Textbooks devote a lot of attention to the important dynamical concepts of acceleration and force, however, following Galileo, the kinematical velocity is defined in coordinate-dependent way, such that the eventual conceptual distinction among absolute and relative velocities, among non-relativistic and relativistic velocities, is obscure in the expression like this '$x = vt$'. How such 'v' depends on the choice of the reference system? how depends on the choice of mathematical coordinates? What are abstract properties of the set of all relative velocities, including the law of addition-composition of relative velocities?

So, the main question is about the precise axiomatic coordinate-free definition of the concept of relative velocity. The distinct theoretical conceptual definitions of relative velocity must proceeds to the experimental measurements. How to measure, without understanding what concept are you going to measure? Textbooks repeat 'a passenger sitting in a moving train is at rest in relation to the train, but in motion in relation to the ground'. How one can see this in a coordinate expression '$x = vt$'? Where in these three symbols, $\{x, t, v\}$, there is a train, passenger, ground? We need to have something like at least two-variable expression

$$\mathbf{v}(S, \text{passenger}) = \begin{cases} \mathbf{0}_{\text{train}} & \text{if } S \text{ is a train} \\ \neq \mathbf{0} & \text{if } S \text{ is a ground} \end{cases}, \qquad \mathbf{v}(\text{train}, \text{train}) \equiv \mathbf{0}_{\text{train}}. \tag{2.11}$$

From this main point of view of the precise definition of the concept of relative velocity, the Einstein Definition 2.1, is the very precise mathematical definition: the Einstein's relative velocity is coordinate-free, and basis-free. From this Einstein's definition one can deduce many properties of such Einstein's ternary velocities that we are listening in the next Section, and one can easily deduce the coordinate expression for any system of coordinates (we left this to interested students). Reader do not need to like the Einstein definition of the relative velocity, each reader could invent his own different definition of the relative velocity. However, I hope that all readers agree that the Einstein Definition 2.1 is the precise mathematical definition. We do not need at this moment to enter to the experimental problems of the measurements of the Einstein's velocities, nor to the question does the Nature like or dislike the mathematical Einstein's definition?

Lets come back to our conclusion: the Einstein velocity needs the choice of the preferred reference system, if we believe-postulate that there must be one and only one relative velocity among each pair of massive bodies. Einstein's definition say: yes, there is one and only one relative velocity among each pair of massive bodies, *provides* that the preferred reference system (absolute space) was choused. Therefore, the concept of the preferred reference system, an aether, is built in the Einstein velocities.

- Preferred reference system is attractive for explaining the non-symmetrical ageing of twins (Herbert Dingle, Rodgrigo de Abreu, Vasco Guerra, Subhash Kak).

- Preferred reference system is meaning-less for believers in Lorentz-covariance as the corner**stone** of Physics (the Lorentz relativity-group is Sacred!).

- Preferred reference system is a fault for believers in the Relativity Principle (no need for the choice of a preferred reference system in order to have a relative velocity).

Nevertheless one can accept special relativity with Lorentz-relativity-group axiom, admitting that in this case we must violate the relativity principle in order to have the unique Einstein's relative velocity. The uniqueness of relative velocity needs the choice of one reference system to be preferred. Therefore, the special relativity with Lorentz relativity-group transformations, not only is perfectly consistent with a special system of reference (alias preferred system, Einstein's lost frame, aether, fixed stars, etc), as concluded independently in [Rodrigo de Abreu 2004; de Abreu & Guerra 2005; Kak 2007], but, such special relativity does not exists at all without preferred reference system. No choice of a preferred vector P in Definition 2.1, no velocity $\mathbf{v}$ of Bob relative to Alice.

We see the astonishing conflict! The special relativity with Lorentz-relativity-group axiom necessarily contradicts to the Relativity Principle.

Not all reference systems can be considered any more as equivalent! Instead, one reference system necessarily must be chosen in order that each pair of massive bodies has the unique relative velocity.

2.1 Rodrigo de Abreu: restricted Relativity Principle with absolute space

Rodrigo de Abreu in 2002, proposed to reject/abandon the Relativity Principle in favor of 'restricted Relativity Principle' that allows the absolute space with a preferred reference system, referred to as 'the Einstein's lost frame'. This idea was future developed in [Rodrigo de Abreu 2004; Rodrigo de Abreu & Vasco Guerra 2005; Guerra & de Abreu 2006]. The velocity relative to preferred reference system is said to be the *absolute* velocity, and a velocity relative to non-preferred system is said to be the Einstein velocity [Rodrigo de Abreu 2004]. The starting point of Rodrigo de Abreu (also jointly with Vasco Guerra) is the observation that the Einstein synchronization of clocks can be made in one and only one reference system. Analysis of the clock synchronization (related to one-way versus two-ways light velocity) leads Authors to consider the abandoning of the Relativity Principle (that all reference systems are equivalent), not only as the unique alternative, but also as the very natural resolution that is not at all grievous. This is why Rodrigo de Abreu with Vasco Guerra in their recent monograph devote entire Chapter 4 for the historical analysis of the origin of the Relativity Principle, citing Galileo, Poincaré, Feynman, and trying to convince the Readers that the Principle of Relativity is *compatible* with absolute space. All this in order to be consistent with the main de Abreu's axiom: 'clock's synchronization needs a preferred-Einstein's reference frame'.

2.9 Axiom (de Abreu axiom)**.** The one-way light-velocity is source-free in **one and only one** reference system. This reference system is said to be the Einstein lost-frame.

De Abreu and Guerra in their monograph on pages 17-18 are citing Einstein:

> Befindet sich im Punkte A des Raumes eine Uhr, so kann ein in A befindlicher Beobachter die Ereignisse in der unmittelbaren
> ...
> If there is a clock at point A in space, then an observer located

> at A can evaluate the time of events in the immediate vicinity of A ... [Einstein 1905, pp. 893-894]

This citation is interpreted, by Rodrigo de Abreu and Vasco Guerra, to be equivalent to the de Abreu's Axiom 2.9. If Einstein would understand the special relativity as the de Abreu Axiom 2.9, why two pages before, on his pp. 891-892, Einstein is interpreting the Relativity Principle that *no* phenomena are related to the absolute rest, and that the light-velocity is emitter-free (without stressing the uniqueness of the 'rest frame')?

With de Abreu Axiom 2.9, it is very natural to claim that the meaningful physical theory does needs necessarily the choice of preferred reference system. Few citations from Chapter 4 of monograph by Rodrigo de Abreu and Vasco Guerra:

> The principle of relativity is so important to physics that it gave its name to Einstein's theory of relativity. [De Abreu & Guerra 2005, p. 57]

> Very remarkably, Poincaré's principle of relativity is formulated under the assumption of absolute space. [De Abreu & Guerra 2005, p. 63]

> In any case, the crucial idea is that the principle of relativity does not say "all is relative" and it is by no means incompatible with the notion of absolute space.
> [De Abreu & Guerra 2005, p. 65]

The Einstein Definition 2.1 of the ternary relative velocity, implies that *all* these velocities are reciprocal. If $\mathbf{v}$ is the Einstein velocity of Bob relative to Alice, then the Einstein velocity of Alice relative to Bob is $-\mathbf{v}$, for *each* choice of the preferred reference system P. However, this is not the case in de Abreu & Guerra's special relativity theory [de Abreu & Guerra 2005]. De Abreu & Guerra in their monograph introduced three different concepts of velocities. The Abreu-Guerra's *absolute* velocity, denoted here by $\mathbf{v}_{AG}$, is a velocity of reference system, say Bob S', relative to the preferred absolute space at rest (say Aether S), for the chosen event P [Abreu & Guerra 2005, pages 41-42, Figure 3.10],

$$\mathbf{v}_{AG}(P,S,S') \equiv \mathbf{v}_{AG}(\text{event},\text{Aether},\text{Bob}) = \frac{\mathbf{x}}{t} - \left(\frac{t'}{t}\right)^2 \frac{\mathbf{x}'}{t'}. \tag{2.12}$$

This *absolute velocity* is defined in terms of the coordinates of an event P, therefore in fact, it is the 'absolute velocity' of Bob 'as seen by an event P'. It is not obvious that the choice of a spectator-event P must be irrelevant for this definition (2.12) of absolute velocity. One can guess that this 'preferred spectator-event P' (denoted accidently on Figure 3.10 on page 41 by the first letter of 'preferred'), is probably assumed to be in the rest relative to the Aether (or it is in the rest relative to the Bob?). What would be if the event P will be chosen to be neither in the rest relative to Aether, nor in the rest relative to the Bob?

De Abreu and Guerra also define the *Einstein speed* that is the reciprocal Einstein velocity among two frames [Abreu & Guerra 2005, page 74], and this concept seems coincide with the Einstein definition 2.1 (for co-planar systems, see below). Moreover, the Authors have also the 'Rodrigo' non-reciprocal *relative speed* defined on page 44. What seems to me to be the most essential peculiarity of the Abreu & Guerra's special relativity theory, that their absolute-velocity $\mathbf{v}_{AG}$ (2.12), is *not* reciprocal, and the velocity of the Eather as measured by Bob has much higher scalar-magnitude when compared with the absolute velocity of the Bob as measured by the absolute-space-Aether, [Abreu & Guerra 2005, page 42], viz.

$$\mathbf{v}_{AG}(\text{event}, \text{Bob}, \text{Aether}) = -(\gamma_{\mathbf{v}})^2\, \mathbf{v}_{AG}(\text{event}, \text{Aether}, \text{Bob}). \tag{2.13}$$

Therefore, the Abreu-Guerra's special relativity theory, does not accept the democratic interpretation of the Relativity Principle as the Principle that *all* reference systems are equivalent, and, moreover, Abreu-Guerra theory also does not accept the Einstein democratic Definition 2.1 that *all* relative velocities are reciprocal.

2.2 Relativity-groupoid as alternative

Do exists some alternative theory that is completely compatible with the Relativity Principle? The alternative philosophy is to keep Relativity Principle, however, *replace* the Lorentz relativity-group (with choice-dependent Einstein's relative velocity), by a relativity groupoid (with choice-free axiomatic binary relative velocity). The alternative is to consider the relative velocity among massive bodies as the primary concept, and then derive/deduce/*define* the transformation among reference systems in terms of this primordial, given a priori, binary relative velocity.

The Einstein special relativity theory, consider the isometry Lorentz relativity transformation, $L \in O(1,3)$, as the primordial concept, and the relative velocity as the derived concept,

$$\text{relativity transformations} \quad \Longrightarrow \quad \text{the Einstein relative velocities.}$$

The possible alternative is to axiomatize the concept of the unique binary kinematical relative velocity as the primordial concept, and *derive* the relativity transformation among reference systems in terms of this given choice-free binary velocity,

$$\text{relative velocities} \quad \Longrightarrow \quad \text{relativity transformations.}$$

Then, could we have a hope that such *derived* set of all relativity transformations, parameterized by the choice-free axiomatized relative velocities, will coincide with the *group* of Lorentz isometries (parameterized by the choice-dependent Einstein's velocities)?

The aim of the present note is to introduce the binary relative velocity-morphism that is the choice-free, and show that such axiomatic velocity can *not* parameterize the isometric Lorentz transformation. The one reason, among other, is that the domain of the action parameterized by the choice-free velocity is restricted to the two-dimensional sub-manifold of all vector fields. The set of all relativity transformations parameterized by binary relative velocities has the structure of a groupoid (that is not a group), and the addition of binary velocities is associative. This formulation is perfectly consistent with the principle of relativity, because all reference systems are equivalent, and there is no need for the choice of the preferred reference system. The experimental predictions of the groupoid relativity, versus the predictions of the Lorentz relativity-group, are discussed in several other papers [Oziewicz 2006, 2007]. In particular, within the groupoid relativity, the inverse of relative velocity is not reciprocal, however this groupoid non-reciprocity is different from non-reciprocity in Abreu-Guerra theory (2.13),

$$\mathbf{v}^{-1}\cdot\mathbf{v} = \begin{cases} -\ \gamma_{\mathbf{v}}\ \mathbf{v}^2, & |\mathbf{v}^{-1}| = |\mathbf{v}| \quad \text{in groupoid relativity,} \\ -(\gamma_{\mathbf{v}})^2\mathbf{v}^2, & |\mathbf{v}^{-1}| \neq |\mathbf{v}| \quad \text{in Abreu \& Guerra relativity.} \end{cases} \tag{2.14}$$

Within the groupoid relativity, the inverse is an involutiv operation, $(\mathbf{v}^{-1})^{-1} \equiv \mathbf{v}$, whereas this is not the case within Abreu & Guerra theory.

Zaripov consider anisotropic Finsler spacetime, and proved that the Finslerian inverse operation (of relative velocity) is non-reciprocal, $\mathbf{v}^{-1} \neq -\mathbf{v}$, and, in Zaripov's theory, it is also non-involutive, $(\mathbf{v}^{-1})^{-1} \neq \mathbf{v}$, [Zaripov 2006].

3 The addition of Einstein's velocities is non-associative

In 1905 Albert Einstein introduced relativity of simultaneity, and derived the addition of relative velocities parameterized the isometric Lorentz relativity-transformations. The differentials of the Lorentz coordinate transformation gives the $\oplus$-addition of such Einstein's isometric velocities as follows [Fock 1955, 1961 §16, formula (16.08), 1964]

$$\mathbf{v} \oplus \mathbf{u} = \frac{\mathbf{u} + \mathbf{v}}{1 + \mathbf{v} \cdot \mathbf{u}/\mathbf{c}^2} \quad + \quad \frac{\gamma_{\mathbf{u}}}{(\gamma_{\mathbf{u}} + 1)} \frac{\mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{u})}{(\mathbf{c}^2 + \mathbf{v} \cdot \mathbf{u})} \tag{3.1}$$

$$= \frac{\gamma_{\mathbf{u}}\, \mathbf{u} + \mathbf{v}}{\gamma_{\mathbf{u}}\, (1 + \mathbf{v} \cdot \mathbf{u}/\mathbf{c}^2)} + \frac{\gamma_{\mathbf{u}}}{(\gamma_{\mathbf{u}} + 1)} \frac{(\mathbf{v} \cdot \mathbf{u})\, \mathbf{u}}{(\mathbf{c}^2 + \mathbf{v} \cdot \mathbf{u})}, \tag{3.2}$$

$$\Longrightarrow \quad \gamma_{\mathbf{v} \oplus \mathbf{u}} = \gamma_{\mathbf{v}}\, \gamma_{\mathbf{u}} \left(1 - \frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{\mathbf{c}^2} \right).$$

The above two versions of the law of addition, (3.1) and (3.2), are related by the identity (1.5)-(1.6). The first version (3.1) is convenient for the particular case of addition of collinear isometric relative velocities, *i.e.* for $\mathbf{v} \wedge \mathbf{u} = 0$. The second version (3.2) is convenient for the particular case of the addition of perpendicular isometric relative velocities,

$$\mathbf{v} \cdot \mathbf{u} = 0 \qquad \Longrightarrow \qquad \mathbf{v} \oplus \mathbf{u} = \mathbf{u} + \frac{\mathbf{v}}{\gamma_{\mathbf{u}}}. \tag{3.3}$$

3.1 Sommerfeld's identity, and Lobachevsky space

A scalar magnitude of the added velocities (3.2) is well known [Sommerfeld 1909, Silberstein 1914; Jackson 1962, 1975, §11.4, formula (11.34)],

$$(\mathbf{v} \oplus \mathbf{u})^2 = 1 - \frac{1}{\{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1 + \mathbf{u} \cdot \mathbf{v}/c^2)\}^2} \quad \Longleftrightarrow \quad \gamma_{\mathbf{v} \oplus \mathbf{u}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1 + \mathbf{u} \cdot \mathbf{v}/c^2), \tag{3.4}$$

$$\mathbf{v} \oplus \mathbf{u} = \mathbf{0} \quad \Longleftrightarrow \quad \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1 + \mathbf{u} \cdot \mathbf{v}/c^2) = 1 \quad \Longleftrightarrow \quad \mathbf{v} + \mathbf{u} = \mathbf{0}.$$

If $\mathbf{c}$ denotes the limiting velocity, $\gamma_{\mathbf{c}} = \infty$, then (3.4) is showing that

$$\gamma_{\mathbf{c}\oplus\mathbf{v}} = \infty = \gamma_{\mathbf{v}\oplus\mathbf{c}}, \quad (\mathbf{c} \oplus \mathbf{v})^2 = \mathbf{c}^2 = (\mathbf{v} \oplus \mathbf{c})^2. \tag{3.5}$$

The Sommerfeld-Silberstein identity (3.4) is proving that the Einstein $\oplus$-addition (3.1)-(3.2) is an internal binary operation on the Lobachevsky manifold of the Einstein's velocities [Fock 1961, §16, formula (16.11)].

Proof. We need to prove that

$$|\mathbf{u} \cdot \mathbf{v}| \leq \frac{\sqrt{(\gamma_{\mathbf{u}}^2 - 1)(\gamma_{\mathbf{v}}^2 - 1)}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} < 1. \tag{3.6}$$

It is sufficient to show that

$$1 \leq \gamma_{\mathbf{u}}\gamma_{\mathbf{v}} - \sqrt{(\gamma_{\mathbf{u}}^2 - 1)(\gamma_{\mathbf{v}}^2 - 1)} \quad \Longleftrightarrow \quad 0 \leq (\gamma_{\mathbf{u}} - \gamma_{\mathbf{v}})^2. \quad \square$$

3.2 Properties of the addition of the Einstein relative velocities

We are using the name 'absolute' as the synonim of 'observer-free'. In the limit of absolute simultaneity, $\oplus$-addition becomes the Galilean-Newtonian $(+)$-addition of relative velocities. Formally $(+)$-addition is an abelian group, and the set of all relative velocities become vectors of a linear algebra.

Here are three properties of the $\oplus$-addition of the Einstein isometric relative velocities (3.1)-(3.2).

3.2.1 The $\oplus$-inverse

The $\oplus$-inverse is the reciprocal velocity, $\mathbf{u}^{-1} = -\mathbf{u}$, as in the case of absolute time:

$$\mathbf{v} \oplus \mathbf{u} = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{v} + \mathbf{u} = \mathbf{0},$$
$$\text{that is:} \quad \oplus -\text{inverse} \;=\; (+) - \text{inverse}. \tag{3.7}$$

3.2.2 Mocanu paradox

The coincidence of the Galilean $(+)$-inverse and the Einstein $\oplus$-inverse, gives the Mocanu paradox [Mocanu 1985, 1986]: $\oplus$-inverse is $\oplus$-automorphism,

$$(\mathbf{v} \oplus \mathbf{u})^{-1} = (\mathbf{v}^{-1}) \oplus (\mathbf{u}^{-1}) \quad \neq \quad (\mathbf{u}^{-1}) \oplus (\mathbf{v}^{-1}). \tag{3.8}$$

Whereas one would expect that the unary inverse operation is an *anti*-automorphism, $(f \circ g)^{-1} = (g^{-1}) \circ (f^{-1})$.

Figure 1: Addition of Einstein's reciprocal relative velocities: Mocanu paradox

$$\begin{tikzcd}
A \arrow[r, bend left, "\mathbf{u}"] \arrow[rr, bend left=60, "\mathbf{v}\oplus\mathbf{u}"] & B \arrow[l, bend left, "-\mathbf{u}"] \arrow[r, bend left, "\mathbf{v}"] & C \arrow[l, bend left, "-\mathbf{v}"] \arrow[ll, bend left=60, "(-\mathbf{u})\oplus(-\mathbf{v})\neq-\mathbf{v}\oplus\mathbf{u}"]
\end{tikzcd}$$

3.2.3 Ungar's discovery: nonassociativity

In 1988 Ungar discovered that the $\oplus$-addition is non-associative [Ungar 1988, p. 71]. Indeed, one can calculate for the two alternative bracketing (here we put $\mathbf{c}^2 = 1$ for simplicity),

$$\frac{\gamma_{\mathbf{w}\oplus(\mathbf{v}\oplus\mathbf{u})}}{\gamma_{\mathbf{w}}\gamma_{\mathbf{v}}\gamma_{\mathbf{u}}} = 1 + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{u} + \frac{\gamma_{\mathbf{u}}}{\gamma_{\mathbf{u}}+1}(\mathbf{w}\wedge\mathbf{u})\cdot(\mathbf{u}\wedge\mathbf{v}),$$
$$\frac{\gamma_{(\mathbf{w}\oplus\mathbf{v})\oplus\mathbf{u}}}{\gamma_{\mathbf{w}}\gamma_{\mathbf{v}}\gamma_{\mathbf{u}}} = 1 + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{u} + \frac{\gamma_{\mathbf{u}}}{\gamma_{\mathbf{u}}+1}(\mathbf{w}\wedge\mathbf{v})\cdot(\mathbf{v}\wedge\mathbf{u}),$$

$$\{\mathbf{w}\oplus(\mathbf{v}\oplus\mathbf{u})\} \ \wedge \ \{(\mathbf{w}\oplus\mathbf{v})\oplus\mathbf{u}\}$$
$$= A(\mathbf{w}\wedge\mathbf{v}) + B(\mathbf{v}\wedge\mathbf{u}) + C(\mathbf{u}\wedge\mathbf{w}) \ \neq \ 0. \quad (3.9)$$

Thus, not only are these two resulting relative velocities, $\mathbf{w}\oplus(\mathbf{v}\oplus\mathbf{u})$, and $(\mathbf{w}\oplus\mathbf{v})\oplus\mathbf{u}$, non collinear (3.9), but also their differ in their scalar *magnitude*.

3.1 Group structure was postulated in 1905. An analysis of the derivation of the Lorentz group as the group of transformations relating observers, and the velocity $\oplus$-addition (3.1)-(3.2) in [Einstein 1905], reveals that the inverse-velocity property (3.7) was the most important tacit independent assumption used most effectively as an axiom [Einstein 1905 p. 901], and is not related to the verbal Einstein's two postulates [1905 pp. 891-892]. The reciprocal-velocity axiom (3.7) tells that every observer measuring some velocity can measure also inverse of this velocity. It is however true that Einstein's reciprocal-velocity axiom (3.7) is absolutely necessary for the derivation of the Lorentz group as the one that relates two observers:

Lorentz group relating observers $\Longrightarrow$ $\{\oplus\text{-inverse} = (+)\text{-inverse}\}$.

3.2 Thomas rotation. The Thomas rotation (Thomas in 1926), means non-transitivity of the parallelism of the spatial frames. The addition of Einstein relative velocities, being non-associative, is not a group operation. It is a *loop* operation [Ungar 1998, 2001; Sabinin & Miheev 1993; Sabinin, Sabinina & Sbitneva 1998; Sbitneva 2001]. Non-associative $\oplus$-addition is counterintuitive and paradoxical: for a system of four or more bodies the $\oplus$-addition of three non-collinear relative velocities gives the **two** distinct velocities between two bodies. There have been attempts [Ungar 2001] to explain the non-associativity, and also Mocanu paradox, as the Thomas rotation (Thomas in 1926), *i.e.* as non-transitivity of the parallelism of the spatial frames. We consider this attempt not satisfactory. Jackson [1962] argued that the Thomas rotation is *necessary* in order to explain factor '2' in the doublet separation for spin-orbit interaction. Einstein was surprised that Thomas's relativistic 'correction' could give factor '2'. Dirac in 1928 explained the same factor and the correct spin levels in terms of the Clifford algebra and the Dirac equation, without invoking the Thomas rotation. The Dirac equation conceptually ought to be understood in terms of the Clifford algebra alone. No longer did anyone need Thomas's precession except for the non-associative $\oplus$-addition of velocities.

Many opponents disagree with non-associative law of addition of the Einstein relative velocities. Opponents claim that if a group G is associative (think about composition of matrices), then the Lobachevski factor space G/H must be necessarily an associative group, even if H is not a normal subgroup, e.g. [John Barrett 2006; Daniel Sudarsky, UNAM, ICN, private communication in 2004]. The rotation group $O(3)$ is not normal subgroup of the Lorentz group $O(1,3)$.

3.3 Groupoid relativity. Most readers certainly will consider the above properties (i)-(ii)-(iii) of the Lobachevsky manifold of Einstein's velocities, as being very attractive from the point of view of mathematics and physics. Moreover, these properties are consistent with the concept of the absolute preferred space, that explain the twin paradox [Herbert Dingle, Rodrigo de Abreu, Subhash Kak]. Does these attractiveness must forbid the consideration of the alternative theories of special relativity that does not need the absolute space?

Maybe some readers would like to consider the necessity of the absolute space to be a deficient property of such theory? No absolute space, $\Rightarrow$ no

Einstein's relative velocities, $\Rightarrow$ no Einstein's special relativity, $\Rightarrow$ no asymmetric biological ageing of twins.

3.4 Definition (Groupoid category)**.** A category is said to be a *groupoid category*, if and only if every morphism has a two-sided inverse. In particular a *group* is a groupoid one-object-category, with just one object, hence with universal unique neutral element-morphism. A groupoid category is said to be *connected* if there is an arrow joining any two of its objects.

Herein we propose to formulate the physics of relativity in terms of the groupoid category of observers, keeping strictly the most democratic interpretation of the Relativity Principle that *all* reference systems are equivalent. The groupoid relativity starts with the axiomatic definition of the binary relative velocities-morphisms, that are choice-free, Axiom 6.2, and conclude that these relative-velocities can not parameterize the isometric Lorentz transformations. In spite of this, the relativity-groupoid predicts the same time-dilation as the relativity-Lorentz-group, however no material rod contraction. These Lorentz-group-free binary relative velocities possess the associative $\circ$-addition, see Tables 1-2. This associative $\circ$-addition is a trivial corollary that follows from two related concepts: the binary relative velocity is a categorical morphism, and a derived groupoidal boost, that can not be an isometry. In the consequence, a bivector, $\mathbf{u} \wedge (\mathbf{u}^{-1}) \neq 0$, does not vanish. The $\circ$-inverse velocity $\mathbf{u}^{-1}$ is given by an isometric Lorentz boost of the Galilean inverse $-\mathbf{u}$. The axiomatic definition of the choice-free binary relative velocity was reinvented and rediscovered frequently and independently by many authors, see for example [Minkowski 1908; Matolcsi 1993 1.3.7, 1.3.8, 4.2.8; 2001 page 91; Bini et al. 1995 page 2551, formula (2.3)].

3.5 Hestenes's relativity theory. Consider two observers, Alice and Bob, represented by time-like future directed normalized vector fields, say Alice represented by A, $A^2 = -1$, and Bob given by a vector field B with $B^2 = -1$. David Hestenes in 1974 formulated the special relativity within Clifford algebra. The Clifford product of two vectors is sum of scalar factor and a Grassmann bivector

$$AB = A \cdot B + A \wedge B = -\gamma_{\mathbf{v}}(1 + \mathbf{v}). \tag{3.10}$$

Therefore Hestenes postulate that that the relative velocity is the Minkowski bivector, and that a scalar magnitude of relative velocity among *each* pair of observers is given by Minkowski scalar product, *i.e.* $\gamma_{\mathbf{v}} \equiv -A \cdot B$. No one choice of preferred observer P in Definition 2.1, could gives the Hestenes's scalar-magnitude of relative velocity for *all* pairs of reference systems, compare with formula (8.5) below. The Hestenes axiom is not compatible with the Einstein's special relativity where the relative velocity is defined in terms of the Lorentz relativity transformation as given by Definition 2.1,

$$\gamma_{\text{ternary}} = \frac{(P \cdot A)^2 + (P \cdot B)^2 - A \cdot B - 1}{2(P \cdot A)(P \cdot B) + A \cdot B + 1} \quad \neq \quad -A \cdot B = \gamma_{\text{binary}}.$$

The Einstein definition of relative velocity as the parameter of the isometry, Definition 2.1, implies that

$$\gamma_{\mathbf{v}(P,A,B)} = -A \cdot B \quad \Longleftrightarrow \quad P \wedge A \wedge B = 0, \tag{3.11}$$

i.e. if and only if the three-body system is co-planar, $P \wedge A \wedge B = 0$.

The Hestenes axiom can be rephrased equivalently that the relative velocities in Definition 2.1 must be restricted by co-planar exterior observers only. Each given observer P is coplanar for many pairs of massive bodies in mutual motion, however can not be coplanar for *all* pairs.

$$P \wedge (L_{P \wedge \mathbf{v}} A) \wedge A = 0. \tag{3.12}$$

Therefore the Hestenes axiomatic relative velocity can not parameterize the arbitrary Lorentz transformation. The set of relativity transformations parameterized by means of Hestenes's relative-velocities is not the same as the Lorentz group. Therefore the Hestenes relativity is not equivalent to Einstein's special relativity.

Start from the set of all Hestenes's relative velocities. Construct relativity transformations among reference systems parameterized in terms of this set of relative velocities. Do we recover Lorentz group?

4 Notation and terminology

In the following $\mathcal{F}$ denotes an $\mathbb{R}$-algebra of scalar fields on space-time, and $\operatorname{der}\mathcal{F}$ denotes the Lie $\mathcal{F}$-module of $\mathbb{R}$-derivations of a ring $\mathcal{F}$. Moreover g

Table 1: How do you add relative velocities?

Galilean binary velocities	Einstein ternary velocities	relativity groupoid binary velocities
Associative	Nonassociative	Associative
The absolute zero/neutral velocity observer-independent		Zero velocity is observer-dependent
The absolute reciprocal inverse		inverse observer-dependent
Abelian group	Loop = unital quasigroup	Groupoid category Velocity is a morphism

Table 2: What it is the groupoid relativity?

	relativity **group** Lorentz isometry group	groupoid **category** relativity **groupoid**
Transformations among observers morfisms	Lorentz group Isometry group	Groupoid category **Not** isometry
Addition of relative velocities	**Non**-associative	Associative. Binary velocity is a morphism

stands for a tensor field of Lorentzian metric with signature $(-+++)$, and will be considered as $\mathcal{F}$-module map:

$$(\operatorname{der}\mathcal{F}) \xrightarrow{g^*=g} (\operatorname{der}\mathcal{F})^*. \tag{4.1}$$

The names, ‘velocity’ and ‘relative velocity’, is used exclusively for bounded *space*-like vector fields $\in \operatorname{der}\mathcal{F}$,

$$\mathbb{V} \equiv \{\mathbf{v} \in \operatorname{der}\mathcal{F} \mid 0 \leq \mathbf{v}^2 < c^2\}. \tag{4.2}$$

All space-like velocities are denoted by lowercase bold roman characters $\mathbf{u}, \mathbf{v}, \mathbf{w}, \ldots \in \mathbb{V}$. The assumption that exists the finite limiting velocity, *i.e.* $(\gamma_{\mathbf{v}})^2 \left(1 - \frac{\mathbf{v}^2}{c^2}\right) \equiv 1$, can be derived as the corollary of a groupoid approach to the velocity as a groupoid morphism.

The terms *observer, observed, body, laboratory,* are used here as synonymous and exclusively for the time-like future-directed and normalized vector fields $\in \operatorname{der}\mathcal{F}$. The set/category of all observers/observed is denoted by ϖ, $\operatorname{obj}\varpi \equiv \{P \in \operatorname{der}\mathcal{F} | P^2 = -1 \in \mathcal{F}\}$. Objects of this category, that here are synonyms for massive bodies and particles, are denoted by uppercase letters $A, B, P, Q, R, S, \ldots \in \operatorname{obj}\varpi$. The main subject of this note is addition of the *space*-like *velocities.* For this reason a phrase ‘4-velocity’, a synonym for our observer and observed $\in \operatorname{obj}\varpi$, will be avoided as confusing.

4.1 Axiom. For an observer P and a velocity $\mathbf{v}$, the condition $P \cdot \mathbf{v} = 0$ is interpreted as necessary and sufficient for observing $\mathbf{v}$ by P.

In the Einstein-Fock formula (3.2) it is understood implicitly that the velocities $\mathbf{u}$ and $\mathbf{v}$ are space-like and can be measured by time-like preferred observer $P \in \operatorname{der}\mathcal{F}$, $P^2 = -1$, who is orthogonal to them, $P \cdot \mathbf{u} = P \cdot \mathbf{v} = 0$.

Let a space-like velocity $\mathbf{u} \in \mathbb{V}$, be a velocity of a body Q relative to an observer P. Then we display this velocity $\mathbf{u}$ as an actual categorical arrow (morphism) which starts/outgoes at observer P (P is a node of the directed graph), and ends/ingoes at an observed body Q,

$$\begin{tikzcd} \cdots \arrow[r] & P \arrow[r, bend left, "\mathbf{u}"] & Q \arrow[l, bend left, "\mathbf{u}^{-1}"] \arrow[r] & \cdots; \end{tikzcd} \qquad \left(Q = P \quad \Longleftrightarrow \quad \mathbf{u} = \mathbf{0} = \mathbf{u}^{-1}\right), \tag{4.3}$$

$$\begin{aligned} \text{observer of } \mathbf{u} &= P, & \text{observed body with } \mathbf{u} &= Q, \\ \text{observed body with } \mathbf{u}^{-1} &= P, & \text{observer of } \mathbf{u}^{-1} &= Q. \end{aligned}$$

5 Why Lorentz boost contradict with the Relativity Principle?

The group of rotations $O(3)$ is not normal subgroup of the Lorentz group $O(3,1)$. Therefore there is no natural decomposition of the Lorentz transformation as a composition of a rotation and a boost. Every such decomposition, $O(3,1) \ni L = \text{Rotation} \circ \text{Boost}$, depends on an auxiliary choice of a preferred time-like observer P. Lorentz boost needs preferred exterior observer, and this is contradictory to the Relativity Principle.

Let P be an observer and $\mathbf{v}$ be a space-like velocity such that $P \cdot \mathbf{v} = 0$. Let us define the $\mathcal{F}$-module endomorphisms $L_{P\wedge\overline{\mathbf{v}}} \in \operatorname{End}_{\mathcal{F}}(\operatorname{der}\mathcal{F})$ as the polynomial in the following trace-less operator $M_{P\wedge\overline{\mathbf{v}}}$,

$$\overline{\mathbf{v}} \equiv \gamma_{\mathbf{v}} \frac{\mathbf{v}}{c}, \quad M_{P\wedge\overline{\mathbf{v}}} \equiv P \otimes_{\mathcal{F}} (g\overline{\mathbf{v}}) - \overline{\mathbf{v}} \otimes_{\mathcal{F}} (gP) \quad \in \operatorname{End}_{\mathcal{F}}(\operatorname{der}\mathcal{F}), \tag{5.1}$$

$$L_{P\wedge\overline{\mathbf{v}}} \equiv \operatorname{id} + M_{P\wedge\overline{\mathbf{v}}} + \frac{(M_{P\wedge\overline{\mathbf{v}}})^2}{\gamma_{\mathbf{v}} + 1} \tag{5.2}$$

$$\Longrightarrow \quad L_{P\wedge\overline{\mathbf{v}}} \circ L_{-P\wedge\overline{\mathbf{v}}} = \operatorname{id} = L_{-P\wedge\overline{\mathbf{v}}} \circ L_{P\wedge\overline{\mathbf{v}}}, \quad L_{P\wedge\overline{\mathbf{v}}}\, P = \gamma_{\mathbf{v}} \left(P + \frac{\mathbf{v}}{c} \right). \tag{5.3}$$

An endomorphism $L_{P\wedge\overline{\mathbf{v}}}$ leaves invariant the space-like P-dependent 2-plane (no rotation!). Moreover $L_{P\wedge\overline{\mathbf{v}}}$ is a g-isometry, $L_{P\wedge\overline{\mathbf{v}}} \in O_g = O(3,1)$. Thus an endomorphism $L_{P\wedge\overline{\mathbf{v}}}$ is a Lorentz P-boost.

For a given Lorentz transformation $L \in O_g$ and a given preferred (exterior) observer P, a Lorentz P-boost $L_{P\wedge PL}$ is given by (5.3) where $M_{P\wedge\overline{\mathbf{v}}}$ must be replaced by

$$M_{P\wedge LP} \equiv P \otimes gLP - (LP) \otimes gP, \tag{5.4}$$

and the scalar Lorentz factor $\gamma_{\mathbf{v}}$ must be replaced by $-(LP) \cdot P$. Then a P-rotation is given by $R^P_L \equiv (L_{P\wedge LP})^{-1} \circ L$. One can check that $R^P_L P = P$.

The above P-decomposition of the Lorentz transformation $L \in O_g$, as a composition of the P-rotation and a P-boost, $L = L_{P\wedge LP} \circ R^P_L$, one can apply for the composition of two Lorentz P-boosts[1]

$$L_{P\wedge\overline{\mathbf{u}}} \circ L_{P\wedge\overline{\mathbf{v}}} = L_{P\wedge\overline{\mathbf{v}\oplus_P\mathbf{u}}} \circ R^P(\mathbf{u},\mathbf{v}) \quad \in SO(1,3). \tag{5.5}$$

[1] In [Oziewicz 2005] there is a misprint in the definition of the $\oplus$-addition (5.5)-(7.2), the order is reversed. This misprint is corrected here.

The composition of the Lorentz P-boosts is not a Lorentz P-boost, it is a P-boost *up* to the Thomas/Wigner P-rotation $R^P(\mathbf{u},\mathbf{v}) \in SO(3)$.

The discussion of Lorentz boost commonly suppresses the observer-dependence, suggesting (incorrectly) that the Lorentz P-boost $L_{P\wedge\overline{\mathbf{v}}}$ is completely fixed by 'a velocity parameter $\mathbf{v}$', e.g. [Jackson 1962 §11; Ungar 2001 p. 254]. This gives incorrect illusion that the Relativity Principle (all reference systems are equivalent), is in the perfect symbiosis with the Einstein special relativity identified with Definition 2.1. The preferred reference system, hidden in the Einstein special relativity, explain non-symmetric ageing of twins [Herbert Dingle 1962, 1972; Rodrigo de Abreu, since 2002; Subhash Kak 2007].

6 Why boost in groupoid relativity is not isometry?

In this Section we define observer-independent boost that appears to be not isometry. This is because, among other, his domain is restricted only for two-dimensional non-linear sub-manifold of all vector fields.

6.1 Definition (Observers and subject-observed)**.** Each non-zero space-like Minkowski vector, $\mathbf{w}$, $\mathbf{w}^2 > 0$, possess the following pair of two-dimensional sub-manifolds of time-like normalized Minkowski vectors

$$\begin{aligned} O_{\mathbf{w}} &\equiv \{A \in \operatorname{der}\mathcal{F} | A^2 = -1, A \cdot \mathbf{w} = 0\} \subset \{P^2 = -1\}, \\ S_{\mathbf{w}} &\equiv \{B \in \operatorname{der}\mathcal{F} | B^2 = -1, B \cdot \mathbf{w} = \mathbf{w}^2\}. \end{aligned} \tag{6.1}$$

The Minkowski hyperboloid, $O_{\mathbf{0}} \equiv S_{\mathbf{0}} \equiv \{P^2 = -1\}$, was named by Minkowski in 1907 to be the world-surface or the cosmograph [Galison 1979 p. 116; Scott Walter 1999, p. 99].

6.2 Axiom (Binary relative velocity)**.** Let $A \in O_{\mathbf{u}}$, and $\mathbf{u}^2 < c^2$, $\mathbf{w} \equiv \gamma_{\mathbf{u}}\mathbf{u}/c$. Then, and *only* then, $\exists!$ body $B = b_{\mathbf{u}}\, A \equiv \gamma_{\mathbf{u}}(A + \frac{\mathbf{u}}{c}) \in S_{\mathbf{u}}$, moving with a velocity $\mathbf{u}$ *relative* to A.

6.3 Corollary. $\gamma_{\mathbf{u}} = -A \cdot B$. Moreover

$$\begin{gathered}
O_{\mathbf{w}} \ni A \qquad \xrightarrow{b_{\mathbf{w}}} \sqrt{1+\mathbf{w}^2}\, A + \mathbf{w} \equiv B \in S_{\mathbf{w}}, \\
O_{\mathbf{w}} \ni A = \frac{B - \mathbf{w}}{\sqrt{1+\mathbf{w}^2}} \xleftarrow{(b_{\mathbf{w}})^{-1}} \qquad B \in S_{\mathbf{w}},
\end{gathered} \tag{6.2}$$

$$\Longrightarrow \quad A \cdot B = -\sqrt{1+\mathbf{w}^2}, \quad \mathbf{w} = B + (A \cdot B)A, \tag{6.3}$$

$$0 \leq \mathbf{w}^2 = (A \cdot B)^2 - 1. \tag{6.4}$$

6.4 Definition. The space-like relative Minkowski velocity $\mathbf{u}$ (being the Minkowski vector, and not the Minkowski bivector) is said to be *binary*,

$$\frac{\mathbf{u}}{c} \equiv \frac{\varpi(A,B)}{c} = i_{gA} \frac{A \wedge B}{A \cdot B} = \frac{B}{-A \cdot B} - A \in \ker(gA). \tag{6.5}$$

6.5 Corollary. Binary velocity is not reciprocal,

$$\ker(gA) \ni \frac{\varpi(A,B)}{c} \equiv \frac{B}{-A \cdot B} - A \quad \neq \quad -\frac{\varpi(B,A)}{c} \in \ker(gB). \tag{6.6}$$

6.6 Corollary. Binary velocity can not parameterize the isometry,

$$O(1,3) \ni \{L(\mathbf{v})\}^{-1} = L(-\mathbf{v}) \quad \Longleftrightarrow \quad \mathbf{v}^{-1} = -\mathbf{v}, \tag{6.7}$$

$$(\mathbf{v}_{\text{isometric}})^{-1} = -\mathbf{v}_{\text{isometric}} \tag{6.8}$$

$$(\mathbf{v}_{\text{binary}})^{-1} = -\gamma_{\mathbf{v}}\, \mathbf{v} - c\left(\gamma_{\mathbf{v}} - \frac{1}{\gamma_{\mathbf{v}}}\right) A, \quad A \cdot \mathbf{v} = 0. \tag{6.9}$$

Clearly $(b_{\mathbf{u}}\, A)^2 = A^2 = -1$. Axiom 6.2 implies that, $B \cdot \mathbf{u} = \gamma_{\mathbf{u}} - \frac{1}{\gamma_{\mathbf{u}}}$. This Axiom motivates the following two diagrammatical rules for outgoing and ingoing arrows/velocities,

$$A \xrightarrow{\mathbf{u}} \ldots \qquad \text{'out' if and only if} \quad A \cdot \mathbf{u} = 0, \tag{6.10}$$

$$\ldots \xrightarrow{\mathbf{u}} B \qquad \text{'in' if and only if} \quad B \cdot \mathbf{u} = \gamma_{\mathbf{u}} - \tfrac{1}{\gamma_{\mathbf{u}}} \equiv \tfrac{\mathbf{u}^2}{\sqrt{1-\mathbf{u}^2/c^2}}. \tag{6.11}$$

A body B can possess an ingoing velocity $\mathbf{u}$ if and only if $B \cdot \mathbf{u} = \gamma_{\mathbf{u}} - \frac{1}{\gamma_{\mathbf{u}}}$. Then, and only then, the unique laboratory $A = (b_{\mathbf{u}})^{-1}\, B = \frac{B}{\gamma_{\mathbf{u}}} - \mathbf{u}$, exists, such that a body B is moving with a velocity $\mathbf{u}$ *relative* to A.

6.7 Corollary. In contrast to the Lorentz boost $L_{P \wedge \overline{\mathbf{u}}}$, (5.3), whose domain is the entire $\mathcal{F}$-module of all Minkowski vector fields $\operatorname{der} \mathcal{F}$, including not time-like Minkowski vectors, the domain of groupoid boost $b_{\mathbf{u}}$ is not-linear two-dimensional sub-manifold of time-like normalized massive bodies that actually can measure the given space-like velocity $\mathbf{u} \in \mathbb{V}$,

$$\begin{aligned} \operatorname{domain}\{L_{P\wedge\overline{\mathbf{v}}}\} &= \operatorname{der}\mathcal{F} \quad \xrightarrow{\dim_{\mathcal{F}}} 4, \\ \operatorname{domain}\{b_{\mathbf{u}}\} &= O_{\mathbf{u}} \quad \xrightarrow{\dim_{\mathcal{F}}} 2. \end{aligned}$$

For $\mathbf{u} \neq 0$, the co-domain $C_{\mathbf{u}}$ of a morphism $b_{\mathbf{u}}$, has no intersection with the domain $D_{\mathbf{u}}$, $C_{\mathbf{u}} \cap D_{\mathbf{u}} = \emptyset$. Therefore, in particular, $(b_{\mathbf{u}})^2$ is not a morphisms. The morphisms in groupoid category may not be composed. Süveges in 1968, pointed that the groupoid structure, rather then that of a group, arise also naturally in gravity theory in curved manifolds.

6.8 Corollary. The groupoid boost b coincide with the Lorentz P-boost L_P (5.3) when acting on preferred observer only

$$Q \cdot \mathbf{u} = P \cdot \mathbf{u} = 0 \quad \Longrightarrow \quad b_{\mathbf{u}}\, Q = \begin{cases} L_{P\wedge\overline{\mathbf{u}}}\, Q = \gamma_{\mathbf{u}}(\mathbf{u} + Q), & \text{iff } Q = P, \\ \quad \neq L_{P\wedge\overline{\mathbf{u}}} Q, & \text{iff } Q \neq P. \end{cases} \tag{6.12}$$

6.9 Corollary. An exterior-observer-independent boost $b_{\mathbf{u}}$ is not g-isometry.

$$\text{Let:} \quad Q^2 = R^2 = -1, \quad Q \cdot \mathbf{u} = R \cdot \mathbf{u} = 0, \quad \text{and} \quad \gamma_R^Q \equiv -Q \cdot R. \tag{6.13}$$

$$\text{Then:} \quad (b_{\mathbf{u}}Q) \cdot (b_{\mathbf{u}}R) - Q \cdot R \;=\; -(\gamma_R^Q - 1)(\gamma_{\mathbf{u}}^2 - 1) \;\neq\; 0. \tag{6.14}$$

The groupoidal-boost $b_{\mathbf{u}}$, is said also to be Lorentz-group-free.

6.10 Corollary. The relativity theory with non-isometric boosts $\{b_{\mathbf{u}}\}$, do not violate the Lorentz invariance or Lorentz covariance. The concept of Lorentz invariance is not applicable.

6.11 Proper-time. Consider two-body massive system $\{A, B\}$. Let $\mathbf{u} = \mathbf{u}(A, B)$ denotes a relative velocity of B as measured by A. Let, moreover, $\mathbf{u}^{-1} = \mathbf{u}(B, A)$ be a relative velocity of A as measured by B. Bodies mutually moved must possess different simultaneous relations, therefore, the velocity $\mathbf{u}$ and his inverse $\mathbf{u}^{-1}$ are each tangent to a different space-like-spaces, and these two spaces are not parallel. Equivalently, $\mathbf{u}$ and $\mathbf{u}^{-1}$, are in the kernels of the different proper-time differential forms. One can assume that $|\mathbf{u}^{-1}| = |\mathbf{u}|$ (this is the case in the Einstein special relativity, but not the case in Abreu-Guerra theory (2.13)). However, seems that it is not so obvious that the reciprocal-velocity property (3.7) is conceptually consistent with the relativity of proper-time. Does an observer measuring a space-like velocity $\mathbf{u}$, can see (spacetime direction of) *her/his* velocity relative to observed body? Does she/he can measure (the spacetime direction of) the inverse velocity $\mathbf{u}^{-1}$? Can we keep the vanishing bi-vector, $\mathbf{u}^{-1} \wedge \mathbf{u} = 0$, as the axiom that Nature like?

7 Associative addition of binary relative velocities

Consider a system of three bodies $\{A, B, C\}$, Figure 2. Clare C is moving with a binary velocity $\mathbf{v}$ relative to Bob B, and Bob B is moving with a binary velocity $\mathbf{u}$ relative to Alice A. What is the binary velocity of Clare C relative to Alice A?

Figure 2: Three body, $\{A, B, C\}$, in relative motions, $B \cdot \mathbf{u}^{-1} = 0$

$$
\begin{tikzcd}
\text{Alice} \arrow[r, "\mathbf{u}"] \arrow[rr, bend right, "\mathbf{v}\circ\mathbf{u}"'] & \text{Bob} \arrow[r, "\mathbf{v}"] \arrow[l, bend right, "\mathbf{u}^{-1}"'] & \text{Clare}
\end{tikzcd}
$$

We abbreviate 'the addition of binary velocities' (which appears to be associative) to $\circ$-addition. One way to introduce $\circ$-addition is to consider the groupoidal-boost b as an isomorphism from the composition of velocities-morphisms to composition of boosts/maps,

$$O(3,1) \not\ni \quad b_{\mathbf{v}\circ\mathbf{u}} \equiv b_{\mathbf{v}} \circ b_{\mathbf{u}} \tag{7.1}$$

$$\text{whereas} \qquad L_{P\wedge\overline{\mathbf{v}\oplus_P\mathbf{u}}} \neq L_{P\wedge\overline{\mathbf{u}}} \circ L_{P\wedge\overline{\mathbf{v}}} \quad \in O(3,1), \tag{7.2}$$

$$
\begin{aligned}
A \cdot \mathbf{u} = 0 \quad &\Longrightarrow \quad b_{\mathbf{u}} A = \gamma_{\mathbf{u}}(A + \mathbf{u}),\\
A \cdot \mathbf{u} = 0 \ \& \ (b_{\mathbf{u}} A) \cdot \mathbf{v} = 0 \quad &\Longrightarrow \quad b_{\mathbf{v}}(b_{\mathbf{u}} A) = \gamma_{\mathbf{v}}(\mathbf{v} + \gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{u}} A),\\
A \cdot (\mathbf{v} \circ \mathbf{u}) = 0 \quad &\Longrightarrow \quad b_{\mathbf{v}\circ\mathbf{u}} A = \gamma_{\mathbf{v}\circ\mathbf{u}}(A + \mathbf{v} \circ \mathbf{u}),
\end{aligned}
$$

$$\Longrightarrow \quad \mathbf{v} \circ_A \mathbf{u} = \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}\circ\mathbf{u}}}(\mathbf{v} + \gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{u}} A) - A. \tag{7.3}$$

The scalar product of the vector A with formula (7.3), and jointly with Lemma 9.2 below, gives

$$\gamma_{\mathbf{v}\circ\mathbf{u}} = \gamma_{\mathbf{v}} \left(\gamma_{\mathbf{u}} + \frac{\mathbf{v}\cdot\mathbf{u}}{c^2}\right) = \gamma_{\mathbf{v}}\gamma_{\mathbf{u}} \left(1 - \frac{\mathbf{v}\cdot\mathbf{u}^{-1}}{c^2}\right), \tag{7.4}$$

$$\mathbf{c}^2 = 1 \quad \Longrightarrow \quad \mathbf{v}\oplus\mathbf{c} = \mathbf{v}\circ\mathbf{c} = \mathbf{c} \quad \text{and} \quad (\mathbf{c}\oplus\mathbf{v})^2 = (\mathbf{c}\circ\mathbf{v})^2 = 1. \tag{7.5}$$

Note that the space-like vectors possess the Euclidean angle if and only if they have the same time-like source. For example, see Figure 2, Bob B is the source for $\mathbf{v}$ and $\mathbf{u}^{-1}$, $B\cdot\mathbf{v} = 0 = B\cdot\mathbf{u}^{-1}$. In Figure 2 and in Table 3, $A\cdot\mathbf{v} = -\mathbf{v}\cdot\mathbf{u} = \gamma_{\mathbf{u}}\,\mathbf{v}\cdot\mathbf{u}^{-1}$, see Lemma 9.2 below, and [Oziewicz 2005].

Table 3: Associative $\circ$-addition of binary relative velocities, Figure 2. The addition of orthogonal relative velocities, $\mathbf{v}\cdot\mathbf{u}^{-1} = 0$, looks 'the same' for binary and ternary relative velocities [Oziewicz 2005].

$$A\cdot\mathbf{u} = 0, \quad B\cdot\mathbf{v} = 0, \quad B\cdot\mathbf{u}^{-1} = 0, \quad \mathbf{u}^{-1} \neq -\mathbf{u}, \quad \mathbf{v}^{-1} \neq -\mathbf{v} \qquad \Longrightarrow$$

$$\left(1 - \tfrac{\mathbf{v}\cdot\mathbf{u}^{-1}}{c^2}\right)\mathbf{v}\circ\mathbf{u} \quad = \quad \mathbf{u} + \frac{\mathbf{v}}{\gamma_{\mathbf{u}}} \quad + \quad \frac{\mathbf{v}\cdot\mathbf{u}^{-1}}{c}A.$$

$$(\gamma_{\mathbf{u}}^2 - 1)\left(1 - \tfrac{\mathbf{v}\cdot\mathbf{u}^{-1}}{c^2}\right)\mathbf{v}\circ\mathbf{u} = \left\{\gamma_{\mathbf{u}}^2\left(1 - \tfrac{\mathbf{v}\cdot\mathbf{u}^{-1}}{c^2}\right) - 1\right\}\mathbf{u} - \gamma_{\mathbf{u}}\,\mathbf{u}^{-1}\cdot\frac{(\mathbf{v}\wedge\mathbf{u}^{-1})}{c^2}$$

The two versions of the law of addition, Table 3 and Table 4, are related by the identity (1.5)-(1.6).

For comparison Table 4 shows non-associative $\oplus$-addition (3.2), which can be presented in the form analogous to (7.3),

$$\gamma_{\mathbf{v}\oplus\mathbf{u}}\,\mathbf{v}\oplus\mathbf{u} = \gamma_{\mathbf{v}}\mathbf{v} + (\gamma_{\mathbf{v}\oplus\mathbf{u}} + \gamma_{\mathbf{v}})\frac{\gamma_{\mathbf{u}}\mathbf{u}}{\gamma_{\mathbf{u}} + 1}. \tag{7.6}$$

7.1 Warning. Reader must not misleads by short notation in Tables 3-4. In the first row for associative $\circ$-addition, all relative velocities are binary, *i.e.* $\mathbf{v} \equiv \varpi(B, C)$ and $\mathbf{u} \equiv \varpi(A, B)$, as is exactly shown on Figure 2. Contrary to this, in the second row for the non-associative $\oplus$-addition, the same letters, $\mathbf{u}$ and $\mathbf{v}$, denotes the Einstein's isometric *ternary* relative velocities, *i.e.* there

Table 4: Nonassociative $\oplus$-addition of reciprocal ternary relative velocities, no Figure. The addition of orthogonal relative velocities, $\mathbf{v}\cdot\mathbf{u}^{-1}=0$, looks 'the same' for binary and ternary relative velocities [Oziewicz 2005].

$$P\cdot\mathbf{u}=0,\quad P\cdot\mathbf{v}=0,\quad \mathbf{u}^{-1}=-\mathbf{u},\quad \mathbf{v}^{-1}=-\mathbf{v}\qquad\Longrightarrow$$

$$\begin{aligned}\left(1-\tfrac{\mathbf{v}\cdot\mathbf{u}^{-1}}{c^2}\right)\mathbf{v}\oplus_P\mathbf{u} \quad &= \quad \mathbf{u}+\tfrac{\mathbf{v}}{\gamma_{\mathbf{u}}}-\tfrac{\gamma_{\mathbf{u}}}{(\gamma_{\mathbf{u}}+1)}\tfrac{(\mathbf{v}\cdot\mathbf{u}^{-1})}{c^2}\,\mathbf{u}\\ &= \quad \mathbf{u}\,+\,\mathbf{v}\quad+\quad\tfrac{\gamma_{\mathbf{u}}}{\gamma_{\mathbf{u}}+1}\,\mathbf{u}^{-1}\cdot\tfrac{(\mathbf{v}\wedge\mathbf{u}^{-1})}{c^2}\end{aligned}$$

$\mathbf{u}=\mathbf{u}(P,A,B)$ and $\mathbf{v}=\mathbf{v}(P,B,C)$, where the preferred exterior time-like observer P can be chosen arbitrarily, and this exterior observer P is not shown on Figures 1-2.

8 Ternary relative velocity

8.1 Theorem. *The Einstein's isometric ternary relative velocity (parameterizing the Lorentz boost) looks like the kind of 'subtraction' of absolute/binary velocities,*

$$\begin{aligned}\mathbf{u}(P,A,B) \;=\; &\frac{P\cdot(A+B)\{(P\cdot B)\varpi(P,B)-(P\cdot A)\varpi(P,A)\}}{(P\cdot A)^2+(P\cdot B)^2-1-A\cdot B}\\ &\simeq i_{gP}\{P\wedge(B-A)\},\end{aligned}\tag{8.1}$$

$$-B\cdot A=(P\cdot B)(P\cdot A)\left\{1-\frac{\varpi(P,B)\cdot\varpi(P,A)}{c^2}\right\},\tag{8.2}$$

$$\begin{aligned}(P\wedge A\wedge B)^2=(P\cdot A)^2+(P\cdot B)^2-1&\\ -(P\cdot A)^2(P\cdot B)^2\left\{1-\frac{(\varpi(P,B)\cdot\varpi(P,A))^2}{c^4}\right\}&.\end{aligned}\tag{8.3}$$

8.2 Corollary. Consider co-planar system of massive bodies, $P\wedge A\wedge B=0$. In this particular case the above set of expressions (8.1)-(8.3) is reduced to the expression often presented by Rodrigo de Abreu and Vasco Guerra publications,

$$\mathbf{u}(P,A,B)|_{P\wedge A\wedge B=0} \;=\; \frac{\varpi(P,B)-\varpi(P,A)}{1-\varpi(P,B)\cdot\varpi(P,A)/c^2}. \tag{8.4}$$

Proof. The expression of the Einstein's ternary velocity (8.1), in terms of absolute/binary relative velocities, for preferred reference system P that, in general, is not no-co-planar for each pair A and B, is more complicated. The last identity, (8.3), follows from the following determinant,

$$(P\wedge R\wedge S)^2 = \det\begin{pmatrix} P^2 & P\cdot R & P\cdot S\\ R\cdot P & R^2 & R\cdot S\\ S\cdot P & S\cdot R & S^2\end{pmatrix}$$
$$= P^2R^2S^2 + 2(P\cdot R)(R\cdot S)(S\cdot P) - S^2(P\cdot R)^2 - P^2(R\cdot S)^2 - R^2(P\cdot S)^2. \quad \square$$

The expression for non-associative $\oplus$-addition is independent of the choice of the exterior observer P. The same formula holds if instead of $P\cdot\mathbf{u}=0=P\cdot\mathbf{v}$, we will assume that $S\cdot\mathbf{u}=0=S\cdot\mathbf{v}$, for completely arbitrary exterior observer S. Even if we made the particular choice $P=A$, in the second rows in Tables 3-4, this does not means that the letter $\mathbf{u}$ in the first row and in the second row denotes exactly the same physical relative velocity, because the inverse is different. The binary velocity $\mathbf{u}$ in the first row of Tables 3-4 is *not* skew-symmetric function of his arguments, $\mathbf{u}=\varpi(A,B)\neq-\varpi(B,A)$. Whereas the ternary isometric velocity in the second row means always the reciprocal velocity, $\mathbf{u}(P,A,B)=-\mathbf{u}(P,B,A)$, and this must hold also for $P=A$, $\mathbf{u}(A,A,B)=-\mathbf{u}(A,B,A)$. Therefore conceptually, the Einstein's relative velocity parameterizing the isometric Lorentz boost is not the same as the binary relative velocity-morphism, $\mathbf{u}(P,A,B)\neq\varpi(A,B)$, even if numerically these expressions sometimes coincide.

All this means that the notation for the Heaviside-FitzGerald-Lorentz scalar factor, $\gamma_{\mathbf{u}}$, must not be identified in both rows in Tables 3-4. In the first rows, $\gamma_{\mathbf{u}}\equiv -A\cdot B$, whereas in the second rows this factor depends also on exterior observer P,

$$\gamma_{\text{ternary}} = \frac{(P\cdot A)^2+(P\cdot B)^2-A\cdot B-1}{2(P\cdot A)(P\cdot B)+A\cdot B+1} \quad\neq\quad -A\cdot B=\gamma_{\text{binary}}. \tag{8.5}$$

8.3 Theorem. *The magnitudes of the binary and ternary relative velocities coincide, $\gamma(binary) = \gamma(ternary)$, if and only if the three-body system is co-planar,*

$$(P \wedge A \wedge B)^2 = 0.$$

9 Inverse for relative binary velocity

The associative $\circ$-addition of relative binary velocities appears in Thesis [Świerk 1988]. Matolcsi [1993, §4.3], and Bini et al. [1995], derived the addition of relative binary velocities without observing the associativity, and without comparing with non-associative addition of Einstein's ternary reciprocal velocities (3.1)-(3.2). The Matolcsi's form need the following substitution into expression for the composition $\mathbf{v} \circ_A \mathbf{u}$,

$$-\frac{\mathbf{u}^2}{c} A \quad = \quad \mathbf{u} + \frac{\mathbf{u}^{-1}}{\gamma_{\mathbf{u}}}. \tag{9.1}$$

9.1 Proposition (Inverse velocity)**.** *A category of massive bodies is a groupoid category and therefore every body has his own separate zero velocity, i.e.* $\mathbf{v}$ *and* $\mathbf{v}^{-1}$ *'do not commute',*

$$\mathbf{0}_{observed} = \mathbf{v} \circ \mathbf{v}^{-1} \quad \neq \quad \mathbf{v}^{-1} \circ \mathbf{v} = \mathbf{0}_{observer}.$$

The $\circ$-inverse of the binary relative velocity depends on the choice of the internal observer, $\mathbf{v}^{-1} = \mathbf{v}^{-1}(\mathbf{v}, P)$, and possess the following properties:

$$\mathbf{v}^{-1} = -L_{P\wedge\bar{\mathbf{v}}}\,\mathbf{v} \quad \Longrightarrow \quad |\mathbf{v}^{-1}| = |\mathbf{v}|, \qquad \mathbf{v}^{-1} \cdot \mathbf{v} = -\frac{\mathbf{v}^2}{\sqrt{1-\mathbf{v}^2}}, \tag{9.2}$$

$$(\mathbf{v} + \mathbf{v}^{-1})^2 = -2(\gamma_{\mathbf{v}} + 1)\left(1 - \tfrac{1}{\gamma_{\mathbf{v}}}\right)^2 \approx \begin{cases} -(\mathbf{v})^4 & \textit{for} \quad |\mathbf{v}| << 1, \\ -\ 2\gamma_{\mathbf{v}} & \textit{for} \quad |\mathbf{v}| \to 1. \end{cases} \tag{9.3}$$

Proof. Matolcsi observed the equality $\mathbf{v}^{-1} = \{\mathbf{v}(B,C)\}^{-1} = \mathbf{v}(C,B) = -L_{B\wedge C}\,\mathbf{v}(B,C)$, where the particular Lorentz boost is parameterized in terms of the initial B, and the final C, time-like vectors only [Matolcsi 1993 1.3.7, 1.3.8, 4.2.3; 2001 page 91]. See also [Bini, Carini and Jantzen 1995 formula (2.3)]. Fahnline was the first who noted that such boost-link, $L_{B\wedge C}$, is not unique [Fahnline 1982, formulas (15)-(16)-(18)]. The most general boost-link as the complete solution of the Lorentz-boost-link-problem is given in

[Oziewicz 2006]. We have

$$B \wedge C = B \wedge \overline{\mathbf{v}}, \quad \text{and} \quad \mathbf{v}^{-1} = -L_{B\wedge\overline{\mathbf{v}}}\,\mathbf{v} = -\gamma\mathbf{v} - c\left(\gamma - \frac{1}{\gamma}\right)B. \quad \square \tag{9.4}$$

9.2 Lemma (Scalar identity for three-body-system)**.** *Consider three-body-system in* Figure 2*, with binary relative velocities. Then,* $\mathbf{v}\cdot\mathbf{u} = -\gamma_{\mathbf{u}}\,\mathbf{v}\cdot\mathbf{u}^{-1}$*. Hence,* $\gamma_{\mathbf{v}\circ\mathbf{u}} = \gamma_{\mathbf{v}}\gamma_{\mathbf{u}}(1-\mathbf{v}\cdot\mathbf{u}^{-1})$.

Proof. By Proposition 9.1, $\mathbf{u}^{-1} = -L_{A\wedge\overline{\mathbf{u}}}\,\mathbf{u}$. Definition (5.2) or (9.4), gives

$$L_{A\wedge\overline{\mathbf{u}}}\,\frac{\mathbf{u}}{c} \quad = \quad \gamma_{\mathbf{u}}\,\frac{\mathbf{u}}{c} + \frac{\gamma_{\mathbf{u}}^2 - 1}{\gamma_{\mathbf{u}}}\,A. \tag{9.5}$$

Moreover, $\mathbf{v}\cdot B = 0$, imply that, $\mathbf{v}\cdot A = -\mathbf{v}\cdot\mathbf{u}$. All together leads to, $\mathbf{v}\cdot\mathbf{u}^{-1} = -\frac{1}{\gamma_{\mathbf{u}}}\,\mathbf{v}\cdot\mathbf{u}$. $\square$

10 Three body system: collinear motion without Lorentz transformations

10.1 Lemma (Three body system)**.** *Consider three body massive system,* Figure 2*, with binary relative velocities only (no exterior observer). One can verify the following circularly-permuted identities,*

$$\begin{aligned} \mathbf{u}\wedge\mathbf{v}^{-1}\wedge(\mathbf{v}\circ\mathbf{u}) &= \mathbf{u}^{-1}\wedge\mathbf{v}\wedge(\mathbf{v}\circ\mathbf{u})^{-1}, \\ \mathbf{u}\wedge\mathbf{v}^{-1}\wedge(\mathbf{v}\circ\mathbf{u})^{-1} &= \mathbf{u}^{-1}\wedge\mathbf{v}\wedge(\mathbf{v}\circ\mathbf{u}), \end{aligned} \tag{10.1}$$

$$\begin{aligned} \gamma_{\mathbf{v}\circ\mathbf{u}} \quad \tfrac{1}{c}(\mathbf{v}\circ\mathbf{u})\wedge\mathbf{u}\wedge\mathbf{v} &= (\mathbf{v}\cdot\mathbf{u}^{-1}) \quad P\wedge K\wedge L\,, \\ \gamma_{\mathbf{v}} \quad \tfrac{1}{c}\mathbf{v}\wedge(\mathbf{v}\circ\mathbf{u})\wedge\mathbf{u}^{-1} &= (\mathbf{u}\cdot(\mathbf{v}\circ\mathbf{u})) \quad P\wedge K\wedge L\,, \\ \gamma_{\mathbf{u}} \quad \tfrac{1}{c}\mathbf{u}\wedge\mathbf{v}^{-1}\wedge(\mathbf{v}\circ\mathbf{u}) &= (\mathbf{v}^{-1}\cdot(\mathbf{v}\circ\mathbf{u})^{-1}) \quad P\wedge K\wedge L\,. \end{aligned} \tag{10.2}$$

10.2 Definition (Coplanar three-body system)**.** A massive three body system $\{A, B, C\}$, Figure 2, is said to be coplanar if, $A\wedge B\wedge C = 0$.

10.3 Proposition (Collinear motion for binary velocities-morphisms)**.** *A massive three body system* $\{A, B, C\}$*, Figure 2, is in collinear motion if and only if this system is coplanar*

$$A\wedge B\wedge C = 0.$$

Proof. To be the true three body system we must assume that $A \wedge B \neq 0$, $B \wedge C \neq 0$ and $C \wedge A \neq 0$. We must treat every massive body on Figure 2, with binary relative velocities only (no exterior preferred observer), on equal footing. A priori collinearity seen by each body separately amounts to different condition.

Seen by Alice : the motion is collinear if and only if, $\mathbf{u} \wedge (\mathbf{v} \circ \mathbf{u}) = 0$.

Seen by Bob : the motion is collinear if and only if the relative velocity $\mathbf{v}$ and inverse $\mathbf{u}^{-1}$ seen by Bob, are collinear, $\mathbf{v} \wedge \mathbf{u}^{-1} = 0$.

Seen by Clare : the motion is collinear if and only if, $\mathbf{v}^{-1} \wedge (\mathbf{v} \circ \mathbf{u})^{-1} = 0$.

Lemma 10.1 tells that *all* three the above collinear conditions jointly imply necessarily the vanishing of the tri-vector $A \wedge B \wedge C = 0$. One can ask: does exists the collinear motion for $A \wedge B \wedge C \neq 0$? Supposing that $A \wedge B \wedge C \neq 0$ and the collinear motion as seen by any one of these bodies separately, we will arrive to contradiction that at least one of the bi-vectors, $\{A \wedge B, B \wedge C, C \wedge A\}$, must vanish. Therefore the Proposition is the only possibility.

Conversely, Definition 10.2 imply the collinear motion for each observer separately. Namely

$$A \wedge B \wedge C = 0 \quad \Longrightarrow \quad (\gamma_{\mathbf{u}}^2 - 1)L = \gamma_{\mathbf{v}}\gamma_{\mathbf{u}} \frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{c^2} A + \gamma_{\mathbf{v}} \left(\gamma_{\mathbf{u}}^2 - 1 - \gamma_{\mathbf{u}}^2 \frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{c^2} \right) B \tag{10.3}$$

$$\Longrightarrow \quad \mathbf{v} \wedge \mathbf{u}^{-1} = 0, \quad \text{etc.} \tag{10.4}$$

The proposition was proved. □

For the case of non-collinear motion, the composed velocity $\mathbf{v} \circ \mathbf{u}$, needs to be the linear combination of *three* binary velocities, $\mathbf{u}, \mathbf{v}$ and inverse $\mathbf{u}^{-1}$ [Świerk 1988, Matolcsi 1993].

In the case of the collinear motion, the composed velocity $\mathbf{v} \circ \mathbf{u}$, must be linear combination of two velocities, $\{\mathbf{u}, \mathbf{v}\}$ or $\{\mathbf{u}, \mathbf{v}^{-1}\}$ or $\{\mathbf{v}, \mathbf{u}^{-1}\}$, only. For example, for some scalars, a and b, $\mathbf{v} \circ \mathbf{u} = a\mathbf{u} + b\mathbf{v}$. However $P \cdot \mathbf{v} = \gamma_{\mathbf{u}} \mathbf{v} \cdot \mathbf{u}^{-1} \neq 0$, and $b = 0$. Therefore $\mathbf{v} \circ \mathbf{u} = a\mathbf{u}$, and

$$\mathbf{u} \wedge (\mathbf{v} \circ \mathbf{u}) = 0 \quad \Longleftrightarrow \quad c^2(\gamma_{\mathbf{u}}^2 - 1)\, \mathbf{v} \circ \mathbf{u} \; = \; \gamma_{\mathbf{u}}^2(\mathbf{u} \cdot (\mathbf{v} \circ \mathbf{u}))\, \mathbf{u}. \tag{10.5}$$

One can arrive to explicit form of addition using identity (1.5) for $\mathbf{u}^{-1} \cdot (\mathbf{v} \wedge \mathbf{u}^{-1})$, with substitution (9.1),

$$(\gamma_{\mathbf{u}}^2 - 1)\left(1 - \frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{c^2}\right) \mathbf{v} \circ \mathbf{u} = \left(\gamma_{\mathbf{u}}^2 - 1 - \gamma_{\mathbf{u}}^2 \frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{c^2}\right) \mathbf{u} - \gamma_{\mathbf{u}} \mathbf{u}^{-1} \cdot \frac{(\mathbf{v} \wedge \mathbf{u}^{-1})}{c^2}, \tag{10.6}$$

$$\frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{c^2} = \pm \frac{\sqrt{(\gamma_{\mathbf{u}}^2 - 1)(\gamma_{\mathbf{v}}^2 - 1)}}{\gamma_{\mathbf{u}} \gamma_{\mathbf{v}}}. \qquad \text{If} \quad A = B \quad \text{then} \quad \mathbf{v} \circ \mathbf{0}_A = \mathbf{v} = \mathbf{0}_C \circ \mathbf{v}. \tag{10.7}$$

This addition of collinear *binary* velocities (10.5) looks the same as the addition of collinear *ternary* velocities (3.1) [Einstein 1905],

$$\mathbf{u} \wedge \mathbf{v} = 0 \qquad \Longrightarrow \qquad \mathbf{v} \oplus \mathbf{u} \;=\; \frac{\mathbf{u} + \mathbf{v}}{1 + \frac{\mathbf{v} \cdot \mathbf{u}}{c^2}}. \tag{10.8}$$

In particular for collinear binary velocities $\mathbf{v} \wedge \mathbf{u}^{-1} = 0$, the last term in (10.6) vanishes, and we have

$$\lim_{\gamma_{\mathbf{u}} \longrightarrow \infty} (\mathbf{v} \circ \mathbf{u}) = \mathbf{u}. \tag{10.9}$$

11 Aberration of light

Abberation of light in terms of the Lorentz transformation was considered among other by Jackson [1962, 1975 §11.3d]. Gjurchinovski [2006] derived the same aberration as Jackson with 'no use of Lorentz transformation', however using the Lorentz length/rod contraction, that is equivalent to the isometry Lorentz transformation.

Abberation of light in terms of the binary not reciprocal relative velocity was considered by Matolcsi [1993 §4.7]. Let in Figure 2, L be light-like, $L^2 = 0$. The space-like light velocity relative to observer P is

$$\frac{\mathbf{c}_P}{c} \equiv \frac{L}{-L \cdot P} - P \quad \Longrightarrow \quad \left(\frac{\mathbf{c}_P}{c}\right)^2 = 1. \tag{11.1}$$

Therefore the magnitude of the light speed is observer-independent, for every non-inertial observer.

The textbooks teach that 'all inertial observers measure the same speed of light', cf. also with [Matolcsi 1993, §4.7.2; Dvoeglazov & Quintanar González

2006, Conclusions]. Such statement suggest that non-inertial observers would measure observer-dependent speed of light. Corrolary 11.1 is stronger: categorical relativity predict that speed of light is independent of *arbitrary* observer, including *all* non-inertial, rotating and accelerating observers.

Wagh noted that non-detection of the frequency-shift in the laser-interferometry experiment [Braxmaier et al. 2002], indicates that the speed of light in vacuum is independent of whether the system of reference is non-inertial or inertial [Wagh 2006]. Wagh consider that Einstein's gravity field equations are not consistent with such observer-independence of light-velocity (11.1).

The direction of the light propagation is observer-dependent,

$$\frac{\mathbf{c}_P}{c} \equiv \frac{L}{-L \cdot P} - P \quad \neq \quad \frac{\mathbf{c}_K}{c} \equiv \frac{L}{-L \cdot K} - K. \tag{11.2}$$

What are the angles between the direction of the light propagation and the relative *binary* velocity as measured by these two massive bodies P and K ?

$$\begin{aligned} \frac{\mathbf{c}_P \cdot \mathbf{u}}{c^2} &= -\frac{1}{\gamma_{\mathbf{u}}} \frac{L \cdot K}{L \cdot P} + 1 = (\cos\, P) \frac{\sqrt{\gamma^2 - 1}}{\gamma}, \\ \frac{\mathbf{c}_K \cdot \mathbf{u}^{-1}}{c^2} &= -\frac{1}{\gamma_{\mathbf{u}}} \frac{L \cdot P}{L \cdot K} + 1 = (\cos\, K) \frac{\sqrt{\gamma^2 - 1}}{\gamma}, \end{aligned} \tag{11.3}$$

$$\left(1 - \frac{\mathbf{c}_K \cdot \mathbf{u}^{-1}}{c^2}\right)\left(1 - \frac{\mathbf{c}_P \cdot \mathbf{u}}{c^2}\right) = \frac{1}{\gamma_{\mathbf{u}}^2} = 1 - \frac{\mathbf{u}^2}{c^2}, \tag{11.4}$$

$$|\mathbf{u}|\{1 + (\cos\, P)(\cos\, K)\} = c\{(\cos\, P) + (\cos\, K)\}. \tag{11.5}$$

Therefore, if $\mathbf{c}_P \cdot \mathbf{u} = 0$, *i.e.* $\cos\, P = 0$, then $\cos\, K = \frac{|\mathbf{u}|}{c}$. For Matolcsi's expression, [Matolcsi 1993 §4.7.3], one needs the substitution, $\cos\, K \longrightarrow -\cos\, K$, why?

The aberration for the Einstein isometric reciprocal ternary relative velocity, $\mathbf{u} = \mathbf{u}(S, P, K)$, was derived by Jackson [1962, 1975], and by Gjurchinovski [2006].

12 Doppler shift

For Doppler shift we refer to [Schrödinger 1922, Brillouin 1970, Brill 1972], and also to the web page http://www.mathpages.com/rr/s2-04/2-04.htm

In this Section we are showing the the Doppler effect does not distinguish the Lorentz group relativity with restricted Relativity Principle (one reference system distinguished to be in an absolute rest), from groupoid relativity with not restricted Relativity Principle. In both theories the expression for Doppler shift is exactly the same.

A vector field L represent massless light radiation, if it is light-like, $L^2 = 0$. Frequency of a light L, relative to a massive observer P, $P^2 = -1$, is $\nu_P \equiv -P \cdot L$, see e.g. [Brill 1972]. Then

$$\nu_Q = -Q \cdot L = -\gamma_{\mathbf{v}}(P + \mathbf{v}/c) \cdot L = \nu_P \gamma_{\mathbf{v}}(1 - \mathbf{c}_P \cdot \mathbf{v}/c^2)$$
$$= \nu_P\, \gamma_{\mathbf{v}}\{1 - (v/c)\cos P\}. \quad (12.1)$$

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