HOW A TOTAL MASS, EQUIVALENT TO THE GRAVITATIONAL BINDING ENERGY SHOULD BE DUMPED, FROM THE REST MASSES OF TWO BODIES FALLING INTO EACH OTHER?

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ABSTRACT

Previously, based on the energy conservation law *(in the broader sense of the concept of "energy", thus embodying "mass")*, as imposed by the *special theory of relativity*, we had proposed to alter the *rest mass* of a an object of mass m_∞ *(measured at a place free of gravitational field)*, gravitationally bound to a host celestial body of mass $\mathcal{M}_\infty$ *(still measured at a place free of surrounding gravitational field)*, practically infinite as compared to m_∞; accordingly, the rest mass of m m_∞ was to be decreased, as much as the binding energy coming into play, in between the two masses of concern.

This manipulation, together with a *quantum mechanical theorem* we had established *(indicating that if the mass of a wave-like object is decreased by a given amount, its internal energy is concomitantly decreased as much)*, did essentially yield the end results of the general theory of relativity, though through a completely different set up than that of this latter theory.

Through our previous work, for simplicity, we had assumed that $m_\infty << \mathcal{M}_\infty$.

Herein we remove this restriction, and we end up with the relationship about the *total binding energy* $E_B(R)$ coming into play

$$E_B(R) = \frac{m_\infty c_0^2}{\mu}\left[1 - \exp\left(-\mu \frac{G\mathcal{M}_\infty}{c_0^2 R}\right)\right] = \frac{m_\infty^2 c_0^2}{\mu}\left[1 - e^{-\mu\alpha(R)}\right] ,$$

Had the original masses m_∞ and $\mathcal{M}_\infty$ happen to have fallen up to the distance R from each other, along with the definitions

$$\alpha(r) = \frac{G\mathcal{M}_\infty}{c_0^2 r} , \quad \mu = \frac{1 + \frac{m_\infty^2}{\mathcal{M}_\infty^2}}{1 + \frac{m_\infty}{\mathcal{M}_\infty}} ;$$

our result is in conformity with the classical linear momentum conservation law.

INTRODUCTION

Previously, based on the energy conservation law *(in the broader sense of the concept of "energy" embodying "mass"),* as imposed by the special theory of relativity,[1] we have proposed to alter the *rest mass* of an object of mass m_∞ *(measured at infinity, in a space free of gravitational field),* gravitationally bound to a host celestial body of mass $\mathcal{M}_\infty$ *(also measured in a space free of gravitational field),*, and this, as much as the binding energy that comes into play, in between the two masses.[2]

In calculating the binding energy E_B, we had assumed that the *host celestial body* of mass $\mathcal{M}_\infty$, is *very massive,* so that through the binding process it does not practically move. This made that, we retrieved a mass equivalent to the binding energy E_B, from the *light object* originally of mass m_∞, *only.*

In what follows, we allow both objects to move, whilst getting bound.

Thus, below we first calculate the *binding energy* of two gravitating bodies, taking into account the *mass deficits* of the objects in consideration, delineated through the process of binding (Section 1). This enables us to determine the *overall relativistic masses* of the two gravitating objects, including the mass deficits *(due to binding)* (Section 2). Next, we derive the general equations of motion via differentiating these relativistic masses *(pointing to the total energies of the objects of concern, thus)* to be held constant on a given orbit (Section 3). This yields us to the classical momentum conservation law, in harmony with our depart point (Section 4). A conclusion is presented thereafter (Section 5).

1. TOTAL GRAVITATIONAL BINDING ENERGY OF TWO OBJECTS, EMBODYING THE MASS DEFICITS DELINEATED THROUGH THE PROCESS OF BINDING

Suppose we set the *light object* of mass m_∞, and the *massive object* of mass $\mathcal{M}_\infty$ *(originally assumed at rest),* simultaneously free *(in the reference system of the distant observer),* at a very big distance from each other. Because of the *Newtonian gravitational force (we had adopted, though for static masses only),* they will accelerate toward each other, and at a given distance from each other, they will acquire, respectively the velocities v and V, thus *the kinetic energies* K_m and K_M. Now, if somehow the original masses m_∞ and $\mathcal{M}_\infty$, are stopped; *masses* respectively equivalent to their kinetic energies are to be retrieved from their internal energies.

If we retain the *conservation of the linear momentum,* we can write

$$\mathcal{M}_\infty V = m_\infty v\,; \tag{1}$$

here we kept the original masses at infinity, since according to our approach the kinetic energies are respectively fueled by the mass deficiencies coming into play, making that the overall masses remain constant through the motion, and equal to their original contents; we will elaborate on this below.

Eq.(1), more essentially, points to an other problem. It is indeed true that a priori, we do not know whether the *total linear momentum* of the system in consideration, is conserved or not, within the frame of our approach. Classically, it is conserved *(owing to Newton's Second Law of Motion).* But the *Newtonian attraction force* $F_{dynamic}$, *between moving objects,* does not behave exactly like the *Newtonian attraction force* F_{static},[3] *reigning between static masses;* F_{static} is in fact greater than $F_{dynamic}$;[2] more precisely, we had shown that the *inertial mass* is greater than the *gravitational mass,* by a factor of γ^2 *(where* γ *is the usual Lorentz dilation factor).* Nonetheless, there appears an easy way of showing that the conservation of linear momentum law should further hold, within the frame of our approach. The force driving the motion between m_∞ and $\mathcal{M}_\infty$, can indeed be seen from both the side of m_∞, and that of $\mathcal{M}_\infty$. Thus, *through the motion,* no matter how the Newtonian attraction force exerted by m_∞ on $\mathcal{M}_\infty$, and the Newtonian attraction force exerted by $\mathcal{M}_\infty$ on m_∞, should be modified, depending on their respective velocities; these forces, should remain equal to each other. Both of these forces, on the other hand, are equal to the rate of changes of the respective *linear momenta.* Thence these latter quantities, should also remain equal to each other, all the way, through the fall. Below, we will elaborate on this too.

Note on the other hand that, Eq.(1) should be written for *allover relativistic masses,* coming into play. Yet through a free fall, the motion is fueled by the transformation of *(for current cases)* a given *(minimal)* part of the mass of the given object into kinetic energy. This makes that the overall masses of the objects falling into each other, are their respective masses, at infinity. That is, Eq.(1), is well rigorous.

For small velocities, this equation makes that the *ratio* f of the *kinetic energy* of m_∞, to the *summation of kinetic energies coming into play,* turns out to be

$$f = \frac{K_m}{K_M + K_m} = \frac{\frac{m_\infty v^2}{2}}{\frac{\mathcal{M}_\infty V^2}{2} + \frac{m_\infty v^2}{2}} = \frac{\mathcal{M}_\infty}{\mathcal{M}_\infty + m_\infty} \quad . \tag{2}$$

(fraction of mass to be retrieved from the small mass) .

This constitutes the *fraction* of the mass equivalent to the binding energy of m_∞ and $\mathcal{M}_\infty$, *to be retrieved from the small mass* m_∞.

The *ratio* F of the *kinetic energy* of $\mathcal{M}_\infty$, to the *summation of kinetic energies* in question, likewise, becomes

$$F = \frac{K_M}{K_M + K_m} = \frac{\frac{\mathcal{M}_\infty v^2}{2}}{\frac{\mathcal{M}_\infty V^2}{2} + \frac{m_\infty v^2}{2}} = \frac{m_\infty}{\mathcal{M}_\infty + m_\infty} = 1 - f \quad . \tag{3}$$

(fraction of mass to be retrieved from the big mass)

This constitutes the *fraction* of mass equivalent to the overall binding energy of m_∞ and $\mathcal{M}_\infty$, *to be retrieved from the big mass* $\mathcal{M}_\infty$.

At relativistic velocities, Eq.(1) must be considered, along with the relativistic relationships of momenta.

Through the free fall, based on our previous work,[2] one can write the *following overall relativistic energy conservation equations, regarding the relativistic masses* $m_{0\gamma}(r_0)$, and $\mathcal{M}_{0\gamma}(r_0)$, of the objects in hand. These ought to be constant, since the kinetic energies they acquire on the way, are to by fueled by the transformation of a minimal part of their respective rest masses:

$$m_{0\gamma}(r_0)c_0^2 = m_{0\infty}c_0^2 \frac{1 - f\,\dfrac{E_B}{m_{0\infty}c_0^2}}{\sqrt{1-\dfrac{v_0^2}{c_0^2}}} = m_{0\infty}c_0^2 \ , \tag{4-a}$$

$$\mathcal{M}_{0\gamma}(r_0)c_0^2 = \mathcal{M}_{0\infty}c_0^2 \frac{1 - F\,\dfrac{E_B}{\mathcal{M}_{0\infty}c_0^2}}{\sqrt{1-\dfrac{V_0^2}{c_0^2}}} = \mathcal{M}_{0\infty}c_0^2 \ , \tag{4-b}$$

E_B is the total binding energy coming into play at the given distance of the two bodies, falling onto each other. Thus $m_\infty\left(1-f\dfrac{E_B}{m_\infty c_0^2}\right)$ and $\mathcal{M}_\infty\left(1-F\dfrac{E_B}{\mathcal{M}_\infty c_0^2}\right)$ are the instantaneous rest masses, at the given respective locations.

This makes that

$$\frac{1 - f\,\dfrac{E_B}{m_{0\infty}c_0^2}}{\sqrt{1-\dfrac{v_0^2}{c_0^2}}} = 1 \ , \tag{4-c}$$

and

$$\frac{1 - F\,\dfrac{E_B}{\mathcal{M}_{0\infty}c_0^2}}{\sqrt{1-\dfrac{V_0^2}{c_0^2}}} = 1 \ . \tag{4-d}$$

Let us define the instantaneous relativistic momenta p_m and p_M, of the two objects:

$$p_m = m_\gamma v = m_\infty v, \tag{4-e}$$

and

$$p_M = \mathcal{M}_\gamma v = \mathcal{M}_\infty V . \tag{4-f}$$

We can then write

$$p_m^2 c_0^2 + m_\infty^2 \left(1 - f\,\frac{E_B}{mc_0^2}\right)^2 c_0^4 = m_\infty^2 c_0^4 \ , \tag{4-g}$$

$$p_M^2 c_0^2 + \mathcal{M}_\infty^2 \left(1 - F\,\frac{E_B}{\mathcal{M}_\infty\, c_0^2}\right)^2 c_0^4 = \mathcal{M}_\infty^2 c_0^4 \ . \tag{4-h}$$

The two momenta, must be equal throughout:

$$m_\infty^2 \left[1 - \left(1 - f\,\frac{E_B}{m_\infty c_0^2}\right)^2\right] = \mathcal{M}_\infty^2 \left[1 - \left(1 - F\,\frac{E_B}{\mathcal{M}_\infty\, c_0^2}\right)^2\right] \ . \tag{5-a}$$

This yields

$$\begin{aligned} m_\infty^2 \left(2f\,\frac{E_B}{m_\infty c_0^2} - f^2\,\frac{E_B^2}{m_\infty^2 c_0^4}\right) &= \mathcal{M}_\infty^2 \left(2F\frac{E_B}{\mathcal{M}_\infty c_0^2} - F^2\,\frac{E_B^2}{\mathcal{M}_\infty^2 c_0^4}\right) \\ &= \mathcal{M}_\infty^2 \left(2(1-f)\frac{E_B}{\mathcal{M}_\infty c_0^2} - (1-f)^2\,\frac{E_B^2}{\mathcal{M}_\infty^2 c_0^4}\right) \ . \\ &= \mathcal{M}_\infty^2 \left(2\frac{E_B}{\mathcal{M}_\infty c_0^2} - 2f\,\frac{E_B}{\mathcal{M}_\infty c_0^2} - \frac{E_B^2}{\mathcal{M}_\infty^2 c_0^4} + 2f\,\frac{E_B^2}{\mathcal{M}_\infty^2 c_0^4} - f^2\,\frac{E_B^2}{\mathcal{M}_\infty^2 c_0^4}\right) \end{aligned} \tag{5-b}$$

or

$$2f\,\frac{E_B}{c_0^2}\left[m_\infty + \mathcal{M}_\infty\left(1 - \frac{E_B}{\mathcal{M}_\infty c_0^2}\right)\right] = \mathcal{M}_\infty\,\frac{E_B}{c_0^2}\left(2 - \frac{E_B}{\mathcal{M}_\infty c_0^2}\right) . \tag{5-c}$$

This yields .

$$f = \frac{2\mathcal{M}_\infty - \dfrac{E_B}{c_0^2}}{2\left(\mathcal{M}_\infty + m_\infty - \dfrac{E_B}{c_0^2}\right)} = \left(\frac{\mathcal{M}_\infty}{\mathcal{M}_\infty + m_\infty}\right)\left[\frac{1 - \dfrac{E_B}{2\mathcal{M}_\infty\, c_0^2}}{1 - \dfrac{E_B}{(\mathcal{M}_\infty + m_\infty)c_0^2}}\right] \tag{6}$$

leading well to Eqs. (2) and (3), were v and V are small, as compared to the speed of light, i.e. the terms in E_B / c_0^2 can be dropped.

Note yet that, in the general case, f depends on the binding energy coming into play.

When the two masses have fallen up to distance r, from each other, their binding energy, $E_B(r)$, based on Newton's attraction force reigning between static masses, as assessed by a distant observer, can be written as

$$E_B(r) = \int_r^\infty \frac{G}{r'^2}\mathcal{M}_\infty\left[1 - F\frac{E_B(r')}{\mathcal{M}_\infty\, c_0^2}\right] m_\infty\left[1 - f\,\frac{E_B(r')}{m_\infty c_0^2}\right]dr' \ . \tag{7-a}$$

$$E_B(r)=\int_r^\infty \frac{G}{r'^2}\left[\frac{\mathcal{M}_\infty c_0^2-F\,E_B(r')}{c_0^2}\right]\left[\frac{m_\infty c_0^2-f\,E_B(r')}{c_0^2}\right]dr'\ . \tag{7-b}$$

This leads to the differential equation

$$-\frac{dE_B(r)}{dr}+G(f\mathcal{M}_\infty+Fm_\infty)\frac{E_B(r)}{r^2c_0^2}-G\,f\,F\frac{E_B^2(r)}{r^2c_0^4}=\frac{G\mathcal{M}_\infty m_\infty}{r^2}\ . \tag{8}$$

Note that F, out of Eq.(6), can be written as

$$F=1-\frac{2\mathcal{M}_\infty-\frac{E_B}{c_0^2}}{2\left(\mathcal{M}_\infty+m_\infty-\frac{E_B}{c_0^2}\right)}=\frac{2m_\infty-\frac{E_B}{c_0^2}}{2\left(\mathcal{M}_\infty+m_\infty-\frac{E_B}{c_0^2}\right)}=\left(\frac{m_\infty}{\mathcal{M}_\infty+m_\infty}\right)\left[\frac{1-\frac{E_B}{2m_\infty c_0^2}}{1-\frac{E_B}{(\mathcal{M}_\infty+m_\infty)c_0^2}}\right]. \tag{8-a}$$

Accordingly $(fm_\infty+F\mathcal{M}_\infty)E_B(r)$ becomes

$$(f\mathcal{M}_\infty+Fm_\infty)E_B(r)=\left\{\left(\frac{\mathcal{M}_\infty^2}{\mathcal{M}_\infty+m_\infty}\right)\left[\frac{1-\frac{E_B}{2\mathcal{M}_\infty c_0^2}}{1-\frac{E_B}{(\mathcal{M}_\infty+m_\infty)c_0^2}}\right]+\left(\frac{m_\infty^2}{\mathcal{M}_\infty+m_\infty}\right)\left[\frac{1-\frac{E_B}{2m_\infty c_0^2}}{1-\frac{E_B}{(\mathcal{M}_\infty+m_\infty)c_0^2}}\right]\right\}E_B(r)\ ,$$

or

(8-b)

$$(f\mathcal{M}_\infty+Fm_\infty)E_B(r)=\left(\frac{\mathcal{M}_\infty^2+m_\infty^2}{\mathcal{M}_\infty+m_\infty}\right)\left[\frac{1-\frac{E_B}{2c_0^2}\frac{(\mathcal{M}_\infty+m_\infty)}{\mathcal{M}_\infty^2+m_\infty^2}}{1-\frac{E_B}{(\mathcal{M}_\infty+m_\infty)c_0^2}}\right]E_B(r)\,. \tag{8-c}$$

This, along with Eq.(8), yields,

$$\begin{aligned}&-\frac{dE_B(r)}{dr}+\frac{G}{r^2c_0^2}\left(\frac{\mathcal{M}_\infty^2+m_\infty^2}{\mathcal{M}_\infty+m_\infty}\right)\left[\frac{1-\frac{E_B}{2c_0^2}\frac{(\mathcal{M}_\infty+m_\infty)}{\mathcal{M}_\infty^2+m_\infty^2}}{1-\frac{E_B}{(\mathcal{M}_\infty+m_\infty)c_0^2}}\right]E_B(r)\\&-\frac{G\mathcal{M}_\infty m_\infty}{(\mathcal{M}_\infty+m_\infty)^2}\left[\frac{1-\frac{E_B}{2\mathcal{M}_\infty c_0^2}}{1-\frac{E_B}{(\mathcal{M}_\infty+m_\infty)c_0^2}}\right]\left[\frac{1-\frac{E_B}{2m_\infty c_0^2}}{1-\frac{E_B}{(\mathcal{M}_\infty+m_\infty)c_0^2}}\right]\frac{E_B^2(r)}{r^2c_0^4}=\frac{G\mathcal{M}_\infty m_\infty}{r^2}\end{aligned}\ .$$

(8-d)

This is the rigorous differential equation that will furnish the static binding energy.

Note that, via ordering the second term on the RHS of this equation, for small binding energies, one can land at

$$\left(\frac{\mathcal{M}_\infty^2+m_\infty^2}{\mathcal{M}_\infty+m_\infty}\right)\left[\frac{1-\dfrac{E_B}{2c_0^2}\dfrac{(\mathcal{M}_\infty+m_\infty)}{\mathcal{M}_\infty^2+m_\infty^2}}{1-\dfrac{E_B}{(\mathcal{M}_\infty+m_\infty)c_0^2}}\right]E_B(r)=\left(\frac{\mathcal{M}_\infty^2+m_\infty^2}{\mathcal{M}_\infty+m_\infty}\right)\left(1-\frac{E_B}{2c_0^2}\frac{(\mathcal{M}_\infty+m_\infty)}{\mathcal{M}_\infty^2+m_\infty^2}+\frac{E_B}{(\mathcal{M}_\infty+m_\infty)c_0^2}\right)E_B(r)$$

$$=\frac{\mathcal{M}_\infty^2+m_\infty^2}{\mathcal{M}_\infty+m_\infty}E_B(r)-\frac{E_B^2(r)}{c_0^2}\left(\frac{1}{2}-\frac{\mathcal{M}_\infty^2+m_\infty^2}{(\mathcal{M}_\infty+m_\infty)^2}\right)=\frac{\mathcal{M}_\infty^2+m_\infty^2}{\mathcal{M}_\infty+m_\infty}E_B(r)+\frac{E_B^2(r)}{2c_0^2}\left(\frac{(\mathcal{M}_\infty-m_\infty)^2}{(\mathcal{M}_\infty+m_\infty)^2}\right)$$

(8-e)

This unfortunately constitutes a *critical result*, and bearing only an approximated term in $E_B^2(r)$, makes it that, via plugging it into Eq.(8-d), one gets unacceptable solutions.

Hence one should attack the rigorous Eq.(8-d), which does not seem an easy one. Therefore, as a first approximation we will neglect the terms in $E_B^2(r)$.*

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* Note that the *summation* of the terms in $E_B(r)$ and in $E_B^2(r)$ of Eq. (5-a) can be written as

$$G\left(\frac{\mathcal{M}_\infty^2+m_\infty^2}{\mathcal{M}_\infty+m_\infty}\right)\frac{E_B(r)}{r^2c_0^2}\left[1-\frac{\mathcal{M}_\infty m_\infty}{(\mathcal{M}_\infty^2+m_\infty^2)(\mathcal{M}_\infty+m_\infty)}\frac{E_B(r)}{c_0^2}\right]. \quad (i)$$

Let us call A, the quantity in between the brackets. Thus, there appears to be two interesting cases. The first one occurs when $m \ll M$. Then

$$A\cong 1-\frac{m_\infty}{\mathcal{M}_\infty^2}\frac{E_B(r)}{c_0^2}\cong 1, \quad (ii)$$

which makes that we can contentedly neglect the term in $E_B^2(r)$, next to the term in $E_B(r)$.

The second interesting case arises when $m_\infty = M_\infty$. Then

$$A=1-\frac{E_B(r)}{4m_\infty c_0^2}\cdot \quad (iii)$$

The solution [Eq.6)] indicates that when $m_\infty = M_\infty$, μ [of Eq.(8-a)] becomes 1, which makes that

$$E_B(r)=m_\infty c_0^2\left[1-e^{-\alpha(r)}\right]. \quad (iv)$$

An extreme case occurs when $\alpha = 1/2$ *(for, the classical singularity)*. In this case, the binding energy E_B given by Eq.(6), can be used to make an estimate, as a first approximation:

$$E_B = m_\infty c_0^2\left[1-e^{-\alpha)}\right]=m_\infty c_0^2(1-0.5+\frac{0.25}{2}-\frac{0.125}{6})\approx 0.6\, m_\infty c_0^2\,; \quad (v)$$

thus A becomes

$$A=1-\frac{E_B}{4m_\infty c_0^2}\approx 0.85\,, \quad (vi)$$

which happens to be still close to unity, making that under the circumstances in consideration, the term in $E_B^2(r)$ should only be envisaged when the objects are in the vicinity of the classical singularity. Otherwise, the solution delineated by Eq.(6) is very satisfactory.

Then, via neglecting the terms in $E_B^2(r)$, the solution of Eq.(8) is straightforward:

$$E_B(R) = \frac{m_\infty c_0^2}{\mu}\left[1 - \exp\left(-\mu \frac{G\mathcal{M}_\infty}{c_0^2 R}\right)\right] = \frac{m_\infty c_0^2}{\mu}\left[1 - e^{-\mu\alpha(R)}\right] \quad , \tag{9}$$

(binding energy of m *and* $\mathcal{M}$*, at a distance* R *from each other)*

when m_∞ and $\mathcal{M}_\infty$ are at a distance R from each other, with the definitions

$$\alpha = \alpha(r) = \frac{G\mathcal{M}_\infty}{c_0^2 r} \quad , \tag{10}$$

and

$$\mu = \frac{1 + \dfrac{m_\infty^2}{\mathcal{M}_\infty^2}}{1 + \dfrac{m_\infty}{\mathcal{M}_\infty}} \quad , \tag{11-a}$$

so that

$$\mu\alpha = \frac{G}{Rc_0^2}\frac{\mathcal{M}_\infty^2 + m_\infty^2}{\mathcal{M}_\infty + m_\infty} \quad . \tag{11-b}$$

The only difference between Eq.(9), and our previous result, where we supposed $\mathcal{M}_\infty$ fixed; is just μ, *coming into play within the exponantial term*, and $1/\mu$ *dividing the quantity within brackets.*

It can be right away observed that, μ is *unity* when m is very small as compared to $\mathcal{M}_\infty$; but μ is still *unity* when m and $\mathcal{M}_\infty$ are equal to each other; in between, μ draws a *minimum* for $m_\infty / \mathcal{M}_\infty = \sqrt{2} - 1$, for which μ becomes $2(2-\sqrt{2})/2 \cong 0.828$.

Having found this solution one can try to better Eq./8-d) in the following manner.

2. OVERALL RELATIVISTIC MASSES, INCLUDING MASS DEFICITS, DUE TO BINDING

We can now derive the *static masses* of the bound objects. Let us, call them $m(r)$ and M (r), when bound at a distance r from each other.

Thus, via Eqs. (2), (3), and (9) one can write

Note that, according to our approach, were m very big as compared to M; the free fall of m into M, from practically an infinite distance and from rest, yields $e^{-2\alpha} \cong 1 - v^2/c_0^2$, where v is the velocity of m [cf. Eq.(10-a), ahead]. For $\alpha = 1/2$, this yields $v/c_0 \approx 0.8$. Yet, the second order term of the expansion of $\sqrt{1 - v^2/c_0^2}$, still remains about %15 of that of the first order; that is the relative correction to be brought to the non-relativistic expression of the kinetic energy of m, is just that much.

$$m(r)c_0^2 = m_\infty c_0^2\left[1 - f\,\frac{E_B(r)}{m_\infty c_0^2}\right] \cong m_\infty c_0^2\left[1 - \frac{1}{1+\frac{m_\infty^2}{\mathcal{M}_\infty^2}}(1-e^{-\mu\alpha})\right] . \qquad (12\text{-a})$$

[*static mass of* the object m *at a distance* r *from* the object $\mathcal{M}_\infty$]

and,

$$\mathcal{M}(r)c_0^2 = \mathcal{M}_\infty c_0^2\left[1 - F\,\frac{E_B(r)}{\mathcal{M}_\infty c_0^2}\right] \cong \mathcal{M}_\infty c_0^2\left[1 - \frac{m_\infty^2}{\mathcal{M}_\infty^2}\frac{1}{1+\frac{m_\infty^2}{M_\infty^2}}(1-e^{-\mu\alpha})\right] . \qquad (12\text{-b})$$

[*static mass of* the object $\mathcal{M}_\infty$ *at a distance* r *from* the object m]

One can immediately check that $m(r)$ and $\mathcal{M}(r)$ reduce indeed to the same quantity, if $m_\infty = \mathcal{M}_\infty$; if further m_∞ can be neglected as compared to $\mathcal{M}_\infty$, then this latter mass remains unaltered, through the binding process.

Suppose now, m_∞ and $\mathcal{M}_\infty$ are engaged in a given motion, around each other.

In order to formulate this motion, as we did previously,[2] we can imagine that

i) we bring the masses of concern, quasistatically from infinity, at the distance r *from each other*, and then

ii) we assign them, their respective orbit velocities.

The motion of m_∞ and $\mathcal{M}_\infty$ will take place around the center of mass of these.

Let then $v(r_m)$ the velocity of the *small object* with respect to the center of mass; we suppose that this object is situated at a distance r_m from the center of mass. Likewise let $V(r_M)$ the velocity of the big object, still with respect to the center of mass; we suppose that this latter object is situated at a distance r_M from the center of mass.

Through the motion of concern, on the given orbit, the *instantaneous total relativistic energy* of each of the bound masses *(embodying the mass deficit due to binding)*, must remain constant. Or the same, the *overall mass* of each of these, i.e. the *rest mass at infinity* decreased as much as the corresponding portion of the binding energy, but at the same time increased as much as the Lorentz dilation factor, should remain constant.

We can accordingly write, *two constancies* based on the motion of m_∞ and $\mathcal{M}_\infty$, around each other:

$$m_\gamma = m_\infty \frac{\left(1 - f\,\frac{E_B}{m_\infty c_0^2}\right)}{\sqrt{1-\frac{v^2}{c_0^2}}} = \text{Constant} = m_\infty \mathcal{D}_m \cong m_\infty \frac{1 - \frac{1}{1+\frac{m_\infty^2}{\mathcal{M}_\infty^2}}\left(1-e^{-\mu\alpha}\right)}{\sqrt{1-\frac{v^2}{c_0^2}}} \quad , \qquad \text{(13-a)}$$

(overall mass of the object m, *on the orbit)*

and

$$\mathcal{M}_\gamma = \mathcal{M}_\infty \frac{\left(1 - f\,\frac{E_B}{M_\infty c_0^2}\right)}{\sqrt{1-\frac{v^2}{c_0^2}}} = \text{Constant} = \mathcal{M}_\infty \mathcal{D}_M \cong \mathcal{M}_\infty \frac{1 - \frac{m_\infty^2}{\mathcal{M}_\infty^2}\frac{1}{1+\frac{m_\infty^2}{\mathcal{M}_\infty^2}}\left(1-e^{-\mu\alpha}\right)}{\sqrt{1-\frac{V^2}{c_0^2}}} \quad ; \qquad \text{(13-b)}$$

(overall mass of the object $\mathcal{M}$ *on the orbit)*

the approximate expressions point to the non-relativistic cases.

Recall that the constants $\mathcal{D}_m$ and $\mathcal{D}_M$ through a free fall of m and $\mathcal{M}$ into each other, turn out to be unity, since the kinetic energies acquired by the masses on the way, based on our approach, are fueled by the respective mass defects.[2]

3. GENERAL EQUATIONS OF MOTION VIA DIFFERANTIATING THE RELATIVISTIC MASSES: PROOF OF THE CONSERVATION OF LINEAR MOMENTUM

By *differentiating* Eqs (13-a) and (13-b), we arrive at

$$m_\infty \frac{1}{\sqrt{1-\frac{v^2}{c_0^2}}}\, d\!\left(\frac{f\,E_B}{m_\infty c_0^2}\right) = m_\infty \frac{\frac{\frac{v dv}{c_0^2}}{\sqrt{1-\frac{v^2}{c_0^2}}}}{1-\frac{v^2}{c_0^2}}\left(1 - f\,\frac{E_B}{m_\infty c_0^2}\right) \quad , \qquad \text{(14-a)}$$

(rigorous equation leading to the equation of motion of m_∞,
regardless we have a free fall, or not)

$$\mathcal{M}_\infty \frac{1}{\sqrt{1-\frac{V^2}{c_0^2}}} d\left(\frac{FE_B}{\mathcal{M}_\infty c_0^2}\right) = \mathcal{M}_\infty \frac{\frac{\frac{VdV}{c_0^2}}{\sqrt{1-\frac{V^2}{c_0^2}}}}{1-\frac{V^2}{c_0^2}} \left(1 - F\frac{E_B}{m_\infty c_0^2}\right) \quad , \qquad (14\text{-a})$$

(rigorous equation leading to the equation of motion of $\mathcal{M}_\infty$,
regardless we have a free fall, or not)

The *Right Hand Sides* of the above equations, clearly refer to the respective changes of the kinetic energies of m and $\mathcal{M}_\infty$. [Recall that the numerators of Eqs. (13-a) and (13-b) point to the rest masses of m and $\mathcal{M}_\infty$, at the given respective locations.]

Thus the *Right Hand Sides* of the above equations, should come to refer to the changes of the *gravitational energies* induced by the *"effective gravitational forces"* driving the motion of m and $\mathcal{M}_\infty$. The strength of *effective gravitational force* exerted by $\mathcal{M}_\infty$ on m_∞, and the strength of the *effective gravitational force* exerted by m on $\mathcal{M}_\infty$, must be equal *(and this will constitute a very convincing physical tool toward the proof of the linear momentum conservation).*

Thence

$$\frac{1}{\sqrt{1-\frac{v^2}{c_0^2}}} m_\infty \frac{d(f\,E_B)}{dr} = \frac{1}{\sqrt{1-\frac{V^2}{c_0^2}}} \mathcal{M}_\infty \frac{d(FE_B)}{dr} \quad . \qquad (15\text{-a})$$

For a free fall [for which the constants coming into play in Eqs. (13-a) and (13-b), are unity], one can further write

$$m_\infty d(f\,E_B)\left(1 - f\,\frac{E_B}{m_\infty c_0^2}\right) = \mathcal{M}_\infty d(FE_B)\left(1 - F\frac{E_B}{M_\infty c_0^2}\right) \quad . \qquad (15\text{-b})$$

This relationship, together with the previous one, appears to be a fundamental key assuring the momentum conservation.

If the origin of the coordinate system is taken at the *center of mass* of m_∞ and $\mathcal{M}_\infty$, then†

† Because the constants D_m and D_M of Eqs. (13-a) and (13-b) are unity, *in the case of a free fall,* m_γ and M_γ become respectively m_∞ and $\mathcal{M}_\infty$. Thence, Eq.(16) harmoniously yields Eq.(1) *(i.e. the total linear momentum conservation equation),* had one taken the derivative of it, with respect time. The problem here, at the first strike seems though, *"it is not trivial to decide whether we should incorporate the 'static masses', or the 'relativisitic masses', with the definition of the 'center of mass' of these".* In fact, it is question of just a definition; yet defining the *center of mass* along *relativistic masses* [via Eq.(16)], is conformous with *the linear conservation law,* whereas defining it based on just *static masses*, is not.

$$m_\infty r_m = \mathcal{M}_\infty r_M \,, \tag{16}$$

together with

$$r = r_m + r_M \ . \tag{17}$$

Eqs. (16) and (17), yield

$$r_M = \frac{m_\infty}{\mathcal{M}_\infty + m_\infty} r \cong F r \ , \tag{18-a}$$

and

$$r_m = \frac{\mathcal{M}_\infty}{\mathcal{M}_\infty + m_\infty} r \cong f\ r \ ; \tag{18-b}$$

or

$$dr_M = \frac{m_\infty}{\mathcal{M}_\infty + m_\infty} d\,r \cong F d\,r \ , \tag{19-a}$$

and

$$dr_m = \frac{\mathcal{M}_\infty}{\mathcal{M}_\infty + m_\infty} dr \cong f\ dr \ ; \tag{19-b}$$

Hence Eqs. (14-a) and (14-b) can be written as

$$\frac{d(f\,E_B)}{dr}\left(1 - f\,\frac{E_B}{m_\infty c_0^2}\right)\frac{dr}{dr_m} dr_m = m_\infty \frac{dr_m}{dt} dv \ , \tag{20-a}$$

$$\frac{d(F\,E_B)}{dr}\left(1 - F\,\frac{E_B}{\mathcal{M}_\infty c_0^2}\right)\frac{dr}{dr_M} dr_M = \mathcal{M}_\infty \frac{dr_M}{dt} dV \ . \tag{20-b}$$

These equations can be transformed into *vectorial equations*, as usual:[2]

$$\frac{d(f\,E_B)}{dr}\left(1 - f\,\frac{E_B}{m_\infty c_0^2}\right)\left(\frac{\mathcal{M}_\infty + m_\infty}{\mathcal{M}_\infty}\right)\frac{\underline{r}_m}{r_m} = m_\infty \frac{d\underline{v}}{dt} = \frac{d\underline{p}_m}{dt} \ , \tag{21-a}$$

$$\frac{d(F\,E_B)}{dr}\left(1 - F\,\frac{E_B}{\mathcal{M}_\infty c_0^2}\right)\left(\frac{\mathcal{M}_\infty + m_\infty}{m_\infty}\right)\frac{\underline{r}_M}{r_M} = M_\infty \frac{d\underline{V}}{dt} = \frac{d\underline{p}_M}{dt} \ ; \tag{21-b}$$

here we have made use of Eqs. (19-a) and (19-b); v and V should be read as

$$v = \frac{dr_m}{dt} \ , \tag{22-a}$$

and

$$V = \frac{dr_M}{dt} ; \tag{22-b}$$

$\underline{r}_m$ and $\underline{r}_M$ are the outward looking unit vectors.‡

Owing to Eq.(15-b), the left hand sides of Eqs. (21-a) and (21-b) are equal to each other, which makes that we come successfully to verify the equalities of the linear momenta.

Eqs. (21-a) and (21-b) are to be solved for the two unknowns r_m and r_M, via the usual procedure. *(Yet this is not the task we propose to undertake, herein.)*

CONCLUSION

Herein we considered two objects m_∞ and $\mathcal{M}_\infty$ falling into each other, and based on our previous work,[2] we proposed to determine the amount of masses, that should be dumped due to binding, from respectively the rest masses of these objects.

Through our previous work, for simplicity, we had assumed that $m_\infty << \mathcal{M}_\infty$. Herein we removed this restriction, and as a *first approximation,* we end up with

$$E_B(R) = \frac{m_\infty c_0^2}{\mu}\left[1 - \exp\left(-\mu \frac{G\mathcal{M}_\infty}{c_0^2 R}\right)\right] = \frac{m_\infty c_0^2}{\mu}\left[1 - e^{-\mu\alpha(R)}\right] \ , \qquad (9)$$

(binding energy of m_∞ *and* $\mathcal{M}_\infty$ *, at a distance* R *from each other)*

for m_∞ and $\mathcal{M}_\infty$ having fallen to a distance R from each other, with the definitions

$$\alpha = \alpha(r) = \frac{G\mathcal{M}_\infty}{c_0^2 r} \ , \qquad (10)$$

and

$$\mu = \frac{1 + \dfrac{m_\infty^2}{\mathcal{M}_\infty^2}}{1 + \dfrac{m_\infty}{\mathcal{M}_\infty}} \ , \qquad (11\text{-a})$$

The fraction f of E_B, to be taken out of m becomes

‡ Were m_∞ negligible as compared to M_∞, these two equations reduce to

$$\frac{G\mathcal{M}_\infty}{r^2}\left(1 - \frac{v^2}{c_0^2}\right)\frac{\underline{r}_m}{r_m} = \frac{d\underline{v}}{dt} \ , \qquad (\text{i-a})$$

$$\frac{Gm_\infty}{r^2} e^{-\mu\alpha}\left(1 - \frac{V^2}{c_0^2}\right)\frac{\underline{r}_M}{r_M} = \left[1 - \frac{m_\infty^2}{M_\infty^2}\left(1 - e^{-\mu\alpha}\right)\right]\frac{d\underline{V}}{dt} \ ; \qquad (\text{i-b})$$

thus, under the mentioned circumstance, the LHS of the second equation remains negligible as compared to the LHS of the first, yielding the fact that dv/dt remains negligible as compared to dV/dt, or the fact that the big object stays almost at rest, through the motion; accordingly, the first equation turns out to be the same as that we found previously.[1]

$$f = \frac{2\mathcal{M}_\infty - \frac{E_B}{c_0^2}}{2\left(\mathcal{M}_\infty + m_\infty - \frac{E_B}{c_0^2}\right)} = \left(\frac{\mathcal{M}_\infty}{\mathcal{M}_\infty + m_\infty}\right)\left[\frac{1 - \frac{E_B}{2\mathcal{M}_\infty c_0^2}}{1 - \frac{E_B}{(\mathcal{M}_\infty + m_\infty)c_0^2}}\right] , \qquad (6)$$

and, the fraction F of E_B , to be taken out of $\mathcal{M}_\infty$ is (1-f).

The exact solution of Eq.(8), following the plugging of Eq.(6) into Eq.(7-b), to construct a rigorous set up, remains yet to be pinned down, especially, nearby heavily dense objects.

Our finding may point to a clue to the *quest of inflation*, at the beginning of the universe, since *mass*, equivalent to *binding energy*, must have been manufactured *(out of energy)*, through the process of expansion, following the Big Bang, just like mass is transformed into energy through the free fall, we undertook herein.

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