# Newtonian Limit of Einstein's Field Equations Dark Matter and Dark Energy with Einstein's Lambda

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## 1 Abstract

The stress energy momentum tensor with its invariant contracted scalar from Einstein's field equations are used to show that the pressure term they involve is responsible for inducing *additional* mass density above that which is historically thought to be present. This has significant consequences for both the dark matter and dark energy problems. The additional material in the case of normally gravitating material might reasonably be taken to be the missing dark matter required for the stability of galaxies. The negative pressure of the dark energy is shown to induce just the right amount of extra negatively gravitating material to account for its usually assumed equation of state and also confirming the physical dark energy density I have deduced as being present in astrospace in earlier papers.

## 2 Introduction

The work to be described in this paper is an application of the cosmological model introduced in the papers *A Dust Universe Solution to the Dark Energy Problem* [23], *Existence of Negative Gravity Material. Identification of Dark Energy* [24] and *Thermodynamics of a Dust Universe* [32]. All of this work and its applications has its origin in the studies of Einstein's general relativity in the Friedman equations context to be found in references ([16],[22],[21],[20],[19],[18],[4],[23]) and similarly motivated work in references ([10],[9],[8],[7],[5]) and ([12],[13],[14],[15],[7],[25],[3]). The applications can be found in ([23],[24],[32],[36],[34][40]). Other useful sources of information are ([17],[3],[30],[27],[29],[28]) with the measurement essentials coming from references ([1],[2],[11],[37]). Further references will be mentioned as necessary.

## 3 Einstein's Lambda

Einstein presented his field equation for general relativity in 1915, ten years after the publication of his *special* theory of relativity which, as its name implies, was all about observers views from frames of reference in relative motion and this replaced the precursor *Newton's Theory of dynamics.* The big jump from the special theory to the general theory was that gravitation was incorporated into the simple space time geometrical structure of the special theory to produce a much more involved space time geometry. Thus general relativity replaced *Newton's Theory of Gravity*. However, you look at these developments one thing is certain, the evolution of theory must involve the more advanced forms reducing to the less advanced forms under some approximation conditions. Only if theory develops under such a constraint, can we be sure that the evolution process is consistent and built on sound foundations. There are other important constraints on theory evolution, the most important of which is theoretical agreement with measurement. Here we consider a case of the first mentioned, *limiting consistency with earlier theory*. In particular, I shall examine an important aspect of the extensively studied Newtonian limit of general relativity. This limit is discussed in detail in all the serious textbooks on general relativity with correct mathematical and physical analysis. However, it seems to me that they all miss important consequences that can be seen from a careful and possibly alternative interpretation of the physics of that limit. One of these missed consequences involves the Lambda term and its involved pressure and the other missed consequence is about the normal mass density contribution and its involvement with pressure. The first issue is the question of the theoretical *amount* of physical dark energy density that is present as the vacuum state of three space. This aspect is studied in this paper using the Einstein field equations together with the Friedman equations rather than as in earlier work when I used the Friedman equations exclusively. The second missed consequence could supply a general relativistic explanation for the missing *dark matter* problem and essentially falls out from the first $\Lambda$ issue.

Einstein's fundamental and brilliant idea in writing down his field equations was to make a connection between physical conditions in space time as given by the stress energy momentum tensor, $T_{\mu\nu}$, with a geometrical interpretation of these conditions as given by the geometrical structure terms on the left hand side of the equation. See reference (4.1) . Thus the right hand side is essentially input with the left hand side describing an output geometrical pattern explanation of the input physical structure on the right. Clearly, however, the introduction of the Lambda term, (4.2), does complicate this simple input-output relationship idea because mathematically the Lambda term could be transposed to the right hand side and then be regarded as actually a contribution to the physical input rather than to the geometry. Either side will do for this term in its raw form as $\pm g_{\mu\nu}\Lambda$ with, of course, the appropriate sign. However, if it is put on the right hand side there is a temptation to interpret it as being the consequence of a physical space density additional to the physical input tensor $-\kappa T_{\mu\nu}$ and of the same form such as $-\kappa T_{0,\mu\nu}$ but with the extra 0 subscript to distinguish it. $\Lambda$ would then acquire a simple physical interpretation in terms of an additional energy density $\rho_0$, the (44) component of $T_{0,\mu\nu}$. Consequently we would get

$$- g_{\mu\nu}\Lambda = -\kappa T_{0,\mu\nu}. \tag{3.1}$$

The (44) term of $T_{\mu\nu}$ is $c^2\rho$ and the (44) term of $g_{\mu\nu}$ is $= 1$ so that (3.1) would imply

$$\Lambda = \kappa T_{0,44} = \kappa c^2 \rho_0 \tag{3.2}$$

and with $\kappa$ as usual identified as

$$\kappa = 8\pi G/c^4 \tag{3.3}$$

the conclusion is that

$$\rho_0 = \frac{\Lambda c^2}{8\pi G}. \tag{3.4}$$

Readers familiar with this area of study will recognise (3.4) as giving the value of the constant physical density that Einstein actually chose to account for the addition of his Lambda term in his field equations. I think it is important to understand that the above argument leading to the result (3.4) is completely correct if the original question to be answered is taken to be the following. What value of the (44) term of a stress energy momentum tensor on the right is required to correctly represent the newly introduced Lambda term on the left? I expect that Einstein's derivation of (3.4) was finding the answer to the above question in his original reasoning which led to his result. However, a different answer is obtained from a closer study of the field equations if the following alternative question is asked. What is the value of physical energy density in three space implied by the addition of the Einstein Lambda term on the left? This last question will be answered in the following pages.

## 4 Einstein's Field Equations

The first tensor equation that Einstein proposed in 1915 embodying the general theory was,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}. \tag{4.1}$$

Later, 1917, he modified this with the addition of the so called *Lambda*, $\Lambda$, term so that his equation became

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = -\kappa T_{\mu\nu}. \tag{4.2}$$

The right hand side of this equation is the stress energy momentum tensor, $T_{\mu\nu}$ multiplied by a negative constant $-\kappa$ which can be evaluated on the basis of the assumption that Newtonian gravitation theory is a limiting consequence of these equations. The term $R_{\mu\nu}$ is the Ricci curvature tensor and the term $R$ is a scalar invariant obtained from the Ricci tensor. The term $g_{\mu\nu}$ is the metric tensor which in special relativity has the form and components which can be taken to be given by

$$g_{\mu\nu} = diag(-1, -1, -1, +1). \tag{4.3}$$

The stress energy momentum tensor, describing as it does the local conditions of a physical continuous medium of some sort or other, is in general a very complicated mathematical physical object and only in the cases of a limited number of actual physical structures has it been used as a source term in Einstein's field equations to yield a reasonable geometric structure describable by the left hand side of the field equations. One such physical situation that is viable is what is called a perfect fluid which using special relativity and the assumption that such a fluid will have no internal stresses other than just pressure, $P$, $T_{\mu\nu}$ can be expressed in the form

$$T_{\mu\nu} = g_{\mu\sigma} g_{\nu\zeta} \left(\rho + \frac{P}{c^2}\right) \frac{dx^\sigma}{ds} \frac{dx^\zeta}{ds} - P g_{\mu\nu}. \tag{4.4}$$

If the local material described by the $T_{\mu\nu}$ has no 3-dimensionsl macroscopic movement, then the 4-dimensional velocity vectors in the expression (4.4) can be replaced by velocity vectors of the form,

$$\frac{dx^\sigma}{ds} = (0, 0, 0, c). \tag{4.5}$$

to give

$$T_{\mu\nu} = g_{\mu\sigma} g_{\nu\zeta} \left(\rho + \frac{P}{c^2}\right) c^2 - P g_{\mu\nu}. \tag{4.6}$$

Thus we can read off from this that all component with $\nu \neq \mu$ are given by the first entry below as being zero. The elements with $\nu = \mu \neq 4$ are given by the second, third and fourth entries below. The (44) component is give as $c^2\rho$ at the fifth entry below.

$$\begin{aligned}
T_{\mu\nu} &= 0, \ \nu \neq \mu & (4.7)\\
T_{11} &= P & (4.8)\\
T_{22} &= P & (4.9)\\
T_{33} &= P & (4.10)\\
T_{44} &= c^2\rho. & (4.11)
\end{aligned}$$

Thus the stress energy momentum tensor in the limiting situation chosen can be represented by the diagonal matrix

$$T_{\mu\nu} = diag(P, P, P, c^2\rho) \tag{4.12}$$

and the trace of this matrix called T which is the sum of these diagonal elements is the invariant scalar or zero order tensor,

$$T = 3P + c^2\rho = c^2(3P/c^2 + \rho). \tag{4.13}$$

From this we see that in general $T$ involves adding to the energy density an extra energy density $3P$ or equivalently adding to the mass density an extra mass density $3P/c^2$. It is well known that in relativity cosmological pressure is an additional contribution to mass density that modifies the mass density and so makes an extra relativistic contribution to Newton's law of gravitation. Thus an effective physical mass density is generated of amount $\rho_{eff} = 3P/c^2 + \rho$. This is most easily seen from an equation that can be obtained from the Friedman equation for the acceleration field due to density. This equation is

$$\frac{\ddot{r}}{r} = \frac{\Lambda c^2}{3} - \frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) \tag{4.14}$$

and includes a contribution from the Lambda term, $\Lambda c^2$. This is a very important equation in relation to the acceleration due to gravity at radius $r$ and how that depends on the mass density term $\rho$. Clearly, the pressure term $3P/c^2$ adds to the mass density to produce an effective or physical mass density $3P/c^2 + \rho$, the same addition we noted above in the invariant tensor $T$ derived from the full $T_{\mu\nu}$ by contraction. This makes an extra contribution to the acceleration caused by gravity due to the mass density that occurs in Newtonian theory and indeed is a relativistic contribution that has been known about for years. This extra term makes very good sense in the density context and can be explained as follows. Suppose we wish to determine the magnitude of the acceleration $a_g$ due to gravity at some distance from a *galaxy* of total mass $m_g$ using the Newtonian inverse square law formula,

$$a_g = -\frac{m_g G}{r^2}. \tag{4.15}$$

The minus sign in the above incorporates the fact that the acceleration is normally always towards the *source* of gravitational force. This formula will only be useful if we can find a value to give to the mass symbol $m_g$ of the galaxy. However, galaxies are greatly diverse objects and their masses are obviously not just the sum of the masses of their individual component stars and planets. All the components of a galaxy are held together in relative motions or at rest by forces in the very simplest picture. All such forces and motions will modify the simple sum of mass components. A very simplified type picture of a galaxy is a gas or a fluid with interacting particles so that the stress energy tensor used above does fit the bill and reduces our options in general relativity to that form. However, it should be clear that the stress energy momentum tensor can only be an hyper idealisation of the structure it represents, essentially smoothing out all the complications into the two entities density and pressure. Thus we have little choice but to describe a galaxy using a density function and a pressure term. Also we need to recognise that a galaxy will need to have some sort of bounded spherical volume $v_g$, say, within which its mass density will account for all its mass and outside which the galaxy will have no mass. The frame of reference in which we wish to study the acceleration determines the frame of reference in which the source will be taken to be at rest and I think it is clear that in this frame the mass $m_g = \rho_{eff} v_g$ should be called the *rest mass* of the galaxy and as with rest mass in the case of elementary particles, in this frame, the galaxy will have no total overall motion but it can have the local type of random motions that are responsible for pressure as in the gas analogy, the contribution to the pressure term being strictly internal. Clearly the frame invariant $T$ term is just what is needed to emulate the frame invariant rest mass of an elementary particle.

Returning to the equation for acceleration (4.14) we can see that the negative acceleration is reinforced by the additional *positive* pressure term $3P/c^2$. That is to say it helps holds the system together. This aspect prompts me to speculate that such an addition to the mass of a galaxy could be just what is needed to explain *Dark Matter*. This is the extra mass on top of the usually assumed amount of visible mass due to average density calculations or mass density that is required to balance the dynamical book-keeping. However, as we have seen, it would require a non zero pressure which as it stands is not a property of my dust universe model. However, I think having spatially extended regions, galaxies, scattered throughout the universe in which pressure is not zero whilst in the massively larger regions between them in which pressure is zero would be allowable with the overall effect being a dust universe. This is because the term involved in the acceleration formula is that due to normally gravitating material in which extra energy due to pressure would clump together with the density by which it was induced. Clearly this argument does not apply to the $\Lambda$ term because that is self repulsive material that does not clump and so the extra density due to pressure would disperse uniformly throughout space as is its origin density. As far as I know, this explanation for the existence of *dark matter* has not appeared before in spite of the mathematical physical structure concerned in general relativity having been known about for many years, see Rindler on pressure page 395([16]).

It possible to further simplify $T_{\mu\nu}$ by introducing the concept of equation of state for the fluid motion which is an equation expressing the pressure linearly as a function density similar to an equation used in thermodynamics, $P = RT/V$, taking $\rho$ to have the mass density $RT/(Vc^2\omega)$. The standard form for this equation of state as used in cosmology is

$$\frac{P}{c^2} = \rho\omega, \tag{4.16}$$

where $\omega$ is a dimensionless quantity with a numerical value that is determined by the physical character of the system, it can be negative, zero or positive. If the fluid particles are photons it has the value $1/3$. If the fluid particles are what is called dust that is to say they exert no fluid pressure it has the value 0. If the fluid particles exerts a negative pressure it has to have a negative value, $\omega_\Lambda = -1 \implies P = c^2\omega_\Lambda\rho_\lambda$, assuming that density is always positive. Thus in these three cases explained above the pressure $P$ can be replaced in favour of $\rho$ so that the stress energy momentum tensor invariant scalar, T, at (4.13) can be put into the forms listed below. The original formula (4.13) given first then the case with substitution for the pressure in the general case, $P = c^2\rho w$. Next the case with the pressure substitution for the photon case, $\omega = 1/3$. Next with the pressure substitution for the dust case, $P = 0$ and finally the case with pressure substitution for the negative pressure case, $\omega = -1$.

$$\begin{aligned} T &= c^2(3P/c^2 + \rho),\ \mathit{original formula} && (4.17)\\ T_g &= c^2\rho(3\omega + 1),\ \mathit{general} && (4.18)\\ T_p &= 2c^2\rho,\ \mathit{photons} && (4.19)\\ T_d &= c^2\rho,\ \mathit{dust} && (4.20)\\ T_n &= -2c^2\rho,\ \mathit{negative\ pressure}. && (4.21) \end{aligned}$$

Using the original formula at (4.17), equation (4.14) can be expressed as

$$\frac{\ddot{r}}{r} = \frac{\Lambda c^2}{3} - \frac{4\pi G T}{3c^2}. \tag{4.22}$$

From (4.21) it can be seen that we can account for the $\Lambda$ term by making an identification with the appropriate, $T_n$, as exists with the $T$ term as follows

$$\Lambda c^2 = -\frac{4\pi G T_n}{c^2} \quad (4.23)$$

$$= -\frac{4\pi G(-2c^2)\rho}{c^2} \quad (4.24)$$

$$= 8\pi G\rho. \quad (4.25)$$

Clearly $\rho$ above is Einstein's, $\rho_{\Lambda,E} = \Lambda c^2/(8\pi G)$, but in the acceleration formula the Lambda term should be represented as

$$\frac{\Lambda c^2}{3} = \frac{4\pi G \times 2\rho}{3}, \quad (4.26)$$

indicating that twice Einstein's dark energy, $\rho_\Lambda^\dagger = 2\rho_{\Lambda,E}$, is the physical energy density that causes the acceleration. In reference ([23]), I showed that the general relativistic version for acceleration due to gravity for a mass $M_U$ embedded in and permeated by an infinitely extended uniformly field of $\Lambda$ dark energy corresponds exactly to the Newtonian inverse square law form provided, the Lambda density is taken to be $\rho_\lambda^\dagger$. This is easily derivable from (4.22).

$$\frac{\ddot{r}}{r} = 4\pi r^3 G(\rho_\Lambda^\dagger - \rho)/(3r^2) \quad (4.27)$$

$$= M^\dagger{}_\Lambda G/r^2 - M_U G/r^2 \quad (4.28)$$

$$M_\Lambda^\dagger = 4\pi r^3 \rho_\Lambda^\dagger/3 \quad (4.29)$$

$$M_U = 4\pi r^3 \rho/3. \quad (4.30)$$

# 5 Conclusions

Using the physical information source term, $T_{\mu\nu}$, from Einstein's field equations for general relativity and its invariant scalar contraction, $T$, I have reappraised the significance of the pressure variable $P$ that occurs therein. I make use of well known and well established formulae which shows that the cosmological pressure $P$ measures *induced* mass density additional to the mass density that is described by the 44 component of $T_{\mu\nu}$. This is achieved using the well known details of the Newtonian limit of general relativity. In the case of Einstien's Lambda term, I use the same structured argument to show that the negative pressure associated with the lambda term similarly induces an additional density contribution so doubling up to the amount of dark energy to give for the *physically* present amount $\rho^{\dagger}_{\Lambda} = 2\Lambda c^2/(8\pi G)$ or twice Einstein's value. This is equal to the value I have used for dark energy density in the papers in the sequence preceding this paper, there deduced by exclusively using the Friedman equations. I suggest that the increased amount of density implied for the density of normally gravitating material induced by its pressure could account for the missing dark matter an issue that greatly exercises the astronomical community. Both of these additional *dark* mass energy contributions could well have some bearing on the *Pioneer* anomaly problem.

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