Resolution of the Ehrenfest Paradox

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In this article the resolution of the famous Ehrenfest paradox [1] is presented. The paradox relates to a spinning disc and the Special Relativity Theory (SRT) applied to it. The resolution of the paradox is based on the proposition that the paradox results from an incorrect application of SRT to a system that is not in an inertial motion. The centrifugal and the centripetal forces resulting from the rotation are always present and need to be accounted for. Using the previously derived metric for an axially symmetrical space-time the effect of centrifugal and centripetal forces can be correctly included. When this is done no paradox is obtained and it is shown that the spinning disc has flat space-time geometry. The measured data from experiments conducted on such rotating systems are explained by the inertial mass increase as described by SRT.

1. Introduction

There have been many papers published on the resolution of Ehrenfest paradox with various degrees of success and with various conclusions [1]. Most of them are typically aimed at justifying the application of only SRT to this case and the paradox resolution is often obtained by a very contorted reasoning. The paradox results from applying the Lorentz coordinate transformation to a spinning plate whose circumference should contract while the radius should not since the motion of the radius is always perpendicular to the plate's rotating direction. As a result the circumference, according to SRT, is no longer equal to $L_o = 2\pi R$, which leads to a non-flat space-time geometry that is not a domain of SRT. From this consideration it is clear that only the kinematic approach to resolve this problem, as offered by SRT, is not enough. SRT deals with the systems in inertial motion and does not account for the acceleration and inertial forces. In order to resolve the paradox, it is necessary to use the metric from General Relativity Theory (GRT) or use other space-time metrics that describe the non-flat space-time geometry that may be adopted to include the centrifugal and centripetal forces. The well known metric describing the space-time around a centrally gravitating body that has a mass M is the Schwarzschild metric:

$$\begin{aligned} ds^2 &= \left(1 - R_s / r\right)\left(cdt\right)^2 - \left(1 - R_s / r\right)^{-1} dr^2 \\ &-r^2\left(d\vartheta^2 + \sin\vartheta^2 d\phi^2\right) \end{aligned} \tag{1}$$

where $R_s = 2\kappa M / c^2$ is the Schwarzschild radius, κ the gravitational constant, and c the speed of light. However, a new metric has been recently published [2], which is more accurate and more closely describes the reality for this case:

$$ds^2 = e^{2\phi_n/c^2}\left(cdt\right)^2 - dr^2 - r^2 e^{2\phi_n/c^2} d\phi^2 - dz^2 \tag{2}$$

where the parameter ϕ_n is the Newtonian gravitational potential of a mass configuration with an axial symmetry. The coordinate system for this metric is cylindrical with the symmetry axis in the z direction. A brief derivation of metric shown in Eq. (2) is given in the Appendix.

2. The Paradox Resolution

The new metric can now be used to resolve the paradox. An observer placed on the spinning plate circumference and rotating with it observes a centrifugal force. This force is balanced by the mechanical centripetal force of the plate's material to keep the system in a dynamic equilibrium. To resolve the paradox the problem can be divided into two steps:

In the first step, the centripetal force of the disc acting to counter the centrifugal inertial force can be simulated by a special non-rotating gravitational-like force that is pointing inward to the center of the plate. The simulated gravitational-like potential describing this force can then be substituted into Eq. (2). The potential is calculated from the following considerations: for the centrifugal inertial force from the relativistic Newton's law, as observed in the laboratory coordinate system, it holds that:

$$f_{cf} = m_o\omega^2 r / \sqrt{1 - v^2 / c^2} \tag{3}$$

where m_o is the rest mass of a test body located at the circumference of the disc and where it was considered that the inertial mass depends on the velocity as follows:

$$m_i = m_o / \sqrt{1 - v^2 / c^2} \tag{4}$$

For the compensating force of the disc material that is simulated by the gravitational-like force it will also be considered that it depends on the velocity of the observer located at the circumference according to the gravitational mass velocity dependence[3]:

$$m_g = m_o\sqrt{1 - v^2 / c^2} \tag{5}$$

It is important to note here that this relation does not follow the famous Einstein's Weak Equivalence Principle (WEP), $m_i = m_g$.[4] The potential for the simulated gravitational-like centripetal force is found from the force equilibrium condition:

$$\frac{\partial\varphi_n}{\partial r} m_0\sqrt{1 - v^2 / c^2} = m_0\omega^2 r / \sqrt{1 - v^2 / c^2} \tag{6}$$

(For more details about this equation see the note at the end of Appendix). After integration the potential relative to the laboratory coordinate system is equal to:

$$\phi_n = \int_0^r \frac{\omega^2 r dr}{1 - \omega^2 r^2 / c^2} = -\frac{c^2}{2}\ln\left(1 - \frac{\omega^2 r^2}{c^2}\right) \tag{7}$$

where ω is the plate's angular velocity. The force resulting from this potential thus completely images the centrifugal force and fully compensates its effects. From the disc trajectory simulation point of view it should not matter if the compensating force is the gravitational-like acting directly on the individual atom masses or an inter-atomic solid body force generated by the disc material. The mechanical disc expansion is not considered here.

In the second step, by considering again a stationary observer relative to the plate and thus not considering any Coriolis acceleration and also considering that this observer now does not feel any force from the radial acceleration, the standard SRT Lorentz transform can be used for the coordinate transformation from the rotating to the laboratory system without any problems. The metric line element for the curved space-time generating the simulating fictitious gravitational-like centripetal force is obtained by substituting the potential obtained in Eq. (7) into the metric line element introduced in Eq. (2). The result is:

$$ds^2 = \frac{(cdt)^2}{\left(1-\omega^2 r^2/c^2\right)} - dr^2 - \frac{r^2 d\phi^2}{\left(1-\omega^2 r^2/c^2\right)} - dz^2 \tag{8}$$

From this result it is then easily seen that the circumference length as observed by the laboratory observer is:

$$L_i = L_o / \sqrt{1-v^2/c^2} \tag{9}$$

while due to the Lorentz contraction the circumference length is:

$$L_c = L_o \sqrt{1-v^2/c^2} \tag{10}$$

The effects thus precisely cancel each other and no paradox results. The laboratory observer will see the disk periphery not contracted. The similar conclusion is obtained from the metric in Eq. (8) also for time. The simulated centripetal force causes the time contraction while the Lorentz time dilation compensates this effect. The space-time geometry of the rotating disc as viewed by the laboratory observer is flat. It thus seems that all the SRT effects are being compensated for by the curved metric and the only remaining SRT effect that is not compensated for is the inertial mass increase. It is also worth noticing that the speed of light in the angular direction is c relative to the laboratory observer, or $c \pm v$ relative to the observer positioned on the disc circumference. The first integrals of motion derived from the Lagrangian corresponding to the metric in Eq. (8) are as follows:

$$\frac{1}{(1-\omega^2 r^2/c^2)} \frac{dt}{d\tau} = k \tag{11}$$

$$\frac{r^2}{(1-\omega^2 r^2/c^2)} \frac{d\phi}{d\tau} = k\alpha \tag{12}$$

where k and α are the arbitrary constants of integration. Eliminating $d\tau$ from these equations results in the familiar formula for the conservation of angular momentum:

$$r^2 \frac{d\phi}{dt} = \alpha \tag{13}$$

which is one of the well recognized fundamental principles of physics. The obtained results thus lend validity to the classical explanation and the classical formula for the Sagnac effect without any need for SRT theory. These results are also supported by experiments published elsewhere [5]. Finally, the most convincing argument in support of the presented Ehrenfest paradox resolution comes from the GPS data [6]. It is an experimental fact that the time rate measured anywhere on the surface of Earth is the same. Only the small differences in the gravitational potential thus affect the surface located clock rate.

It is now also possible to consider more complex arrangements. For the case of the two discs placed above each other, one rotating, and the other stationary, or both rotating but in opposite directions, there will be no time difference or the circumference length difference observed by any of the observers the stationary or residing on the rotating discs. The observer that returns to the same position after the completion of the full circle on the rotating disc will have his clocks synchronized with the stationary observer or the *counter-rotating observer*.

It is important to note that the standard Schwarzschild metric does not offer similar solution to the Ehrenfest paradox and does not support the conservation of angular momentum as stated in Eq. (13). This is a consequence of the incorrect metric coefficient standing by the angular coordinate. The resolution of the Ehrenfest paradox using the new metric and the different dependencies of the inertial and gravitational masses on velocity thus provide an important additional support for the correctness of these formulas.

The reasoning used in the above derivation can also be reversed and it could be stated as a theorem that in order to avoid the Ehrenfest paradox the metric for the axially symmetric gravitational field has to have a form given in Eq. (2). It is also possible to generalize Eq. (7) for any static space-time metric as follows:

$$\phi_n = \frac{c^2}{2} \ln \sqrt{\frac{g}{g_0}} \tag{14}$$

and generalize the result further by eliminating the gravitational potential:

$$g_{tt}^2 = g / g_0 \tag{15}$$

where g and g_0 are the metric determinants of the metric line element in Eq. (8) with and without rotation. However, the metric determinant g_0 has a slightly different meaning in a general case as has been explained elsewhere: [2]

These derivations, however, do not agree with the Einstein's field equations and the Einstein's WEP, which ultimately casts a significant doubt on the accuracy and correctness of GRT.

There have been many experiments performed in the past in rotating systems to confirm various GRT phenomena, but as is clear from the above explanation only the SRT inertial mass increase, and the effects related to the inertial mass increase such as the absorption line shift in the Mossbauer Fe^{57} effect can be observed [7]. No GRT effects related to the curved space-time geometry can be measured in these experiments. It is also necessary to understand in detail the construction of the particular clock used in the experiments to make sure that it is the time what is measured and not the inertial mass increase.

3. Conclusion

The new space-time metric is used in resolving the Ehrenfest paradox and the related experimental verifications. The metric allows for an inclusion of the centripetal force into the considerations, which compensates for the time dilation and the length contraction effects of SRT. The inertial mass increase, however, is not compensated for, which explains the published experimental results [7]. The resolution of the paradox thus validates the new metric correctness and the different dependencies of inertial and gravitational masses on velocity.

4. Appendix: Metric for the Space-time with Axial Symmetry

A detailed metric derivation for this space-time is available elsewhere. [2] This derivation takes a different approach. The general form of a metric line element for an axially symmetric space-time with the gravitating axis positioned along the z direction is as follows:

$$ds^2 = g_{tt}\left(cdt\right)^2 - g_{rr}dr^2 - g_{\phi\phi}d\phi^2 - g_{zz}dz^2 \tag{A1}$$

For the metric coefficient standing by the z coordinate it is obvious that it must be unity, since there is an unbounded translational symmetry for this axis and no space-time deformation in this direction is therefore possible. In the next steps the remaining metric coefficients will be found. This is best accomplished by first considering a small test body falling from infinity in the radial direction in this space-time. The Lagrangian describing such a motion is equal to:

$$L = g_{tt}\left(\frac{cdt}{d\tau}\right)^2 - g_{rr}\left(\frac{dr}{d\tau}\right)^2 \tag{A2}$$

Since the Lagrangian itself is also the first integral ($L = c^2$), it is simple to derive the following equations:

$$g_{tt}\frac{dt}{d\tau} = 1 \tag{A3}$$

$$g_{tt}g_{rr}\left(\frac{dr}{d\tau}\right)^2 = c^2 - c^2 g_{tt} \tag{A4}$$

Rearranging and differentiating Eq. (A4) with respect to τ results in the following:

$$\frac{d^2r}{d\tau^2} = -\frac{c^2}{2}\frac{\partial}{\partial\phi_n}\left(\frac{1}{g_{rr}} - \frac{1}{g_{rr}g_{tt}}\right)\frac{\partial\phi_n}{\partial r} \tag{A5}$$

In this equation it was assumed that the metric coefficients are functions of the Newton gravitational potential since this is confirmed for large distances by the validity of the Newton's gravitational law. The partial derivative of the bracket must be a contravariant component of a tensor, since both sides of the equation must be of the same type. There are only two possibilities how to satisfy this requirement, the bracket derivative either equal to g^{tt} or to g^{rr}. To find the solutions it is thus convenient to separate the next steps of the derivation into two cases: the first case for $g_{tt}g_{rr} = 1$, written also as $g_{tt} = g^{rr}$, corresponding to spherical coordinates, and the second case for $g_{rr} = g^{rr} = 1$, corresponding to cylindrical coordinates with the result:

$$\frac{d^2r}{d\tau^2} = -\left(\frac{c^2}{2}\frac{1}{g_{tt}}\frac{\partial g_{tt}}{\partial\phi_n}\right)g^{tt}\frac{\partial\phi_n}{\partial r} \tag{A6}$$

In Eq. (A6) the term in the bracket needs to be unity in order to keep the contravariant character of geometric objects on both sides of the equations the same and also to satisfy the equivalence principle where the acceleration equals the force of gravity. It must therefore hold:

$$\frac{c^2}{2}\frac{1}{g_{tt}}\frac{\partial g_{tt}}{\partial\phi_n} = 1 \tag{A7}$$

By integrating this result using the boundary condition at infinity where the potential is zero, the g_{tt} metric coefficient becomes equal to:

$$g_{tt} = e^{2\phi_n/c^2} \tag{A8}$$

It is interesting to note that Eq. (A8) and the condition $g_{rr}g_{tt} = 1$ follow uniquely from the metric in Eq. (A1) once it is assumed that the gravitational field has a potential and that the metric coefficients depend on it.

The metric coefficient standing by the angular coordinate is found by considering a small test body orbital motion in the space-time defined by the following metric line element:

$$ds^2 = g_{tt}\left(cdt\right)^2 - dr^2 - g_{\phi\phi}d\phi^2 - dz^2 \tag{A9}$$

The Lagrangian describing the motion is then:

$$L = g_{tt}\left(\frac{cdt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2 \tag{A10}$$

The first integral corresponding to the angular coordinate is:

$$g_{\phi\phi}\frac{d\phi}{d\tau} = \alpha \tag{A11}$$

where the suitable integration constants was used. It is well known and many times experimentally confirmed that the orbital motion must satisfy the conservation of angular momentum. From Eqs. (A3) and (A11) then follows that:

$$\frac{g_{\phi\phi}}{g_{tt}}\left(\frac{d\phi}{dt}\right) = \alpha \tag{A12}$$

and from this result then also follows that for the angular metric coefficient it is:

$$g_{\phi\phi} = r^2 g_{tt} \tag{A13}$$

since the metric coefficient standing by the radial coordinate is unity. The radial coordinate distance is equal to the radial physical distance in this case. Substituting these results into Eq. (A9) the metric line element used in Eq. (2) is obtained.

Finally a short comment is needed relating to Eq. (6). According to Eq. (A5) the simulated gravitational force equilibrium with the inertial centrifugal force should be more generally written as:

$$\frac{1}{g_{tt}}\frac{\partial\phi_n}{\partial r}m_o\sqrt{g_{tt}}\sqrt{1 - v^2/c^2} = \frac{m_o\omega^2 r}{\sqrt{g_{tt}}\sqrt{1 - v^2/c^2}} \tag{A14}$$

However, all the terms containing g_{tt} cancel out making Eq. (6) correct also. The derivation of these more general formulas for the inertial mass and the gravitational mass dependence on velocity can be found elsewhere. [8]

References

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