Replacement of the Euler Fluid and Navier-Stokes Equations

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The Euler fluid equations, which can be expressed as the vector equation $(-\nabla p/r) - g = (v \bullet \nabla)v + \partial v/\partial t$, are shown to be missing three important terms and to contain a simplistic version of a fourth. Whenever the fluid is significantly affected by an external thrust and by either a gravitational field or a set of gravitational fields, the Euler fluid equations must be replaced by $(-\nabla p/r) + \left[2(f-3)p/3rr_e\right]R_e - \left(M_e G/r_e^2\right)R_e - (F/r) = (v \bullet \nabla)v + (\partial v/\partial t)$, where two of the three missing terms are combined into the first term times the unit vector R_e (which points radially away from the effective center of gravity), 'f' is numerically the degrees of freedom of the fluid, and 'r_e' is the distance from the effective center of gravity to the differential volume. The third missing term, F, is the non-gravitational force. The simplistic gravitational acceleration term, g, is replaced by a more general expression that takes into account the ubiquitous nature of gravity. The Euler fluid equations are used in the derivation of the Navier-Stokes equations, so the foregoing developments cause these equations to change.

Keywords: Euler, Navier-Stokes, fluid equations, fluid motion, atmosphere, atmospheric particle motion, barometric, barometric equation, degrees of freedom, hydrostatic, Newton

1. Introduction

When a new theory extends knowledge beyond an accepted-as-correct existing theory, and it does so by treating as variable one or more factors that were previously assumed to be constant, the new theory must in the limit agree with the existing theory. A good example of the foregoing principle is the Landau and Lifshitz derivation of the Euler fluid equations [3].

The vector derivation provided by Landau and Lifshitz [*Ibidem*] represents those equations as

$$\left(\frac{-\nabla p}{r}\right) + g = (v \bullet \nabla)v + \frac{\partial v}{\partial t}, \tag{1}$$

where ∇ is the vector gradient, p is the fluid pressure, r is the fluid density, g is the vector acceleration due to Earth's gravity, v is fluid velocity, and t is time.

The vector gravitational term, g, is placed in equation (1) by Landau and Lifshitz—presumably so that, as the fluid velocity, v, approaches the limit zero, equation (1) reduces to the then-accepted-as-correct Newtonian [5] barometric equation (2) in vector form.

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$$\nabla p = +r\, g\,, \tag{2}$$

thus conforming to the principle expressed at the beginning of this section.

For a reason not stated by Landau and Lifshitz [4], their subsequent vector derivation of the Navier-Stokes equations does not conform to that principle. They do not insert into their derivation the gravitational vector acceleration g. Possibly this is due to their taking the minimalist view of equations—the user of the equation can add whatever terms are necessary to represent the problem being addressed. Under this view (with which I do not agree), the Navier-Stokes equations are represented in vector form [4] as

$$-\nabla p + h\left(\nabla \bullet \nabla\right)v + \left(V + \frac{1}{3}h\right)\nabla\left(\nabla \bullet v\right) = r\left[\frac{\partial v}{\partial t} + \left(v \bullet \nabla\right)v\right], \tag{3}$$

where h is the *dynamic viscosity*, and V is the *second viscosity*.

Equation (3) is widely used in fluid mechanics, and is a critical equation in some attempts to forecast changes in Earth's atmosphere. As just pointed out, though, there is a problem with this equation. It does not mesh with reality when the limit is taken as the fluid velocity, v, approaches zero throughout the fluid volume. At this limit, the Navier-Stokes vector equation reduces to

$$\nabla p = 0\,, \tag{4}$$

regardless of the possible presence of one or more strong gravitational fields. This situation should be changed. The gradient (along a radial from the center of gravity) of the pressure clearly can not be zero in a planetary atmosphere.

As it happens, equation (3) has another problem in common with equation (1). Newton's barometric equation [5] has been shown to be wrong [1,2]. When the limit is taken as the fluid velocity, *v* approaches zero throughout the fluid volume, both equations (1) and (3) must reduce to Carpenter's [1] barometric equation (5) in vector form.

$$\nabla p = +r\, g + \left[\frac{2(f-3)p}{3r}\right]R = -r\, gR + \left[\frac{2(f-3)p}{3r}\right]R\,, \tag{5}$$

where f is the number of degrees of freedom of the particles of the fluid, g is the magnitude of the vector acceleration due to Earth's gravity, and R is the unit vector (at the differential volume) pointed radially away from the center of gravity.

Neither the Euler fluid equations nor the Navier-Stokes equations address this matter. The purpose of this paper, therefore, is to derive more-correct equations to replace the Euler fluid and Navier-Stokes equations.

2. Theory

Landau and Lifshitz [3] use the vector form of the Euler fluid equations (1) to derive [4] the vector form of the Navier-Stokes equations (3). For that reason, replacement of the Euler fluid equations is herein treated first.

Euler

The development of Landau and Lifshitz [3] is accepted until the point shown in their equation (2.4), which is rearranged as equation (1) of this paper, where they add the vector term g.

$$\left(\frac{-\nabla p}{r}\right) + g = \left(v \bullet \nabla\right)v + \frac{\partial v}{\partial t}\,. \tag{1}$$

Carpenter [1], though, has shown that Newton's disregarding the effect of the fluid outside of Newton's constant cross-sectional area cylinder is a mistake. Carpenter [*Ibidem*] has also shown that Newton's [5] barometric equation (2) omits another important effect, the $v_\perp^2/r$ acceleration (where $v_\perp$ is the particle speed perpendicular to the radial, r, from the center of gravity to the particle) experienced by single-atom and multiple-atom molecules during their thermal motion. Carpenter's [*Ibidem*] barometric equation (5) includes both of these effects.

Incorporating, into equation (1), both effects shown in equation (5) yields

$$\left(\frac{-\nabla p}{r}\right)+ r\,g+\left[\frac{2(f-3)p}{3r}\right]R=\left(\frac{-\nabla p}{r}\right)- r\,gR+\left[\frac{2(f-3)p}{3r}\right]R=\frac{\partial v}{\partial t}+\left(v\bullet\nabla\right)v \qquad (6)$$

Two more aspects need to be addressed. First, a term $(F$, the non-gravitational force$)$ is needed to account for non-gravitational forces. One example is a planetary ionosphere (which is enclosed by a solid or a non-ionized fluid from below, and by the gravitational escape barrier from above, and which is moving relative to the planetary magnetic field); another example is the force experienced by a constrained fluid within an accelerating vehicle, such as a ship on the sea, an aircraft in flight, or a spacecraft under thrust.

The needed term $(-F)$ is added to equation (6), then that equation is rearranged and expressed in vector form as

$$\left(\frac{-\nabla p}{r}\right)- gR+\left[\frac{2(f-3)p}{3\,r\,r}\right]R-\left(\frac{F}{r}\right)=\left(v\bullet\nabla\right)v+\frac{\partial v}{\partial t}. \qquad (7)$$

Instead of reducing to equation (2) when there is no non-gravitational force and when the fluid velocity v approaches zero in the limit in an idealized perfectly spherical Earth gravitational field, equation (1) should reduce to equation (5). Equation (1), therefore, must be temporarily replaced by equation (7).

The second aspect needing to be addressed concerns the ubiquitous nature of gravitational fields in our universe. It must be noted that no exceptions have yet been found for the application of Gauss' Law to gravity at all distances. Yet, not only Newton's [5] barometric equation (2) but also Carpenter's [1] barometric equation (5) and Euler's [3] fluid equations (1) were derived for use only with respect to Earth. Gravity is ubiquitous though, and the gravitational term should actually be expressed as a summation term covering gravity due to all matter in our universe (because there is the possibility of the fluid being sensibly affected by a large {limit unknown} number of masses). When that is done, the term $-gR$ is replaced by $-\sum_1^n M_i \mathrm{G} R_i/r_i^2$, where i ranges from 1 to the total number of mass particles (n) in the universe, M_i is the mass of the ith particle, G is the gravitational constant, R_i is the unit vector pointed radially away from the ith mass particle toward the differential volume of the fluid, and r_i is the radial distance from the ith mass particle to the differential volume of the fluid. The replacement term can also be expressed as

$$M_e \mathrm{G} R_e/r_e^2 \equiv \sum_1^n M_i \mathrm{G} R_i/r_i^2\,, \qquad (8)$$

where M_e (the effective mass of the effective center of gravity), R_e (the effective unit radial vector from the effective center of mass to the differential volume), and r_e (the effective distance from the effective center of gravity to the differential volume) are each defined by equation (8).

Carpenter's barometric equation (5), when corrected, becomes

$$\nabla p=-r\,M_e \mathrm{G} R_e/r_e^2+\left[\frac{2(f-3)p}{3r_e}\right]R_e\,, \qquad (9)$$

The term $\left[2(f-3)p/3rr\right]R$ in equation (7) thus becomes $\left[2(f-3)p/3rr_{\mathrm{e}}\right]R_{\mathrm{e}}$, and equation (7) itself becomes

$$\left(\frac{-\nabla p}{r}\right)+\left[\frac{2(f-3)p}{3rr_{\mathrm{e}}}\right]R_{\mathrm{e}}-\left(\frac{M_{\mathrm{e}}\mathrm{G}}{r_{\mathrm{e}}^{2}}\right)R_{\mathrm{e}}-\left(\frac{F}{r}\right)=\left(v\bullet\nabla\right)v+\frac{\partial v}{\partial t}. \tag{10}$$

Equation (10), therefore, must permanently replace both equations (7) and (1). When the fluid is in the immediate vicinity of a single large body, such as Earth, equation (10) may reduce to equation (7) or the equivalent (where g is replaced by the acceleration value for the large body) because the magnitude of all gravitational terms other than those contained within the large body may be small enough to be ignored. There are also regions in our universe where the total gravitational effects are sufficiently small compared to the other terms in the Euler fluid equations that the new terms introduced herein can be ignored.

Equation (10), therefore, solves the limit-matching problem in the absence of non-gravitational forces by enabling the Euler fluid equations to reduce (in the limit as v approaches zero throughout the fluid volume) to the corrected form of the barometric equation (9).

Equation (10) must, thus, replace the Euler fluid equations represented as equation (1).

Navier-Stokes

Landau and Lifshitz's derivation [4] of the vector form of the Navier-Stokes equations (3) begins on page 44. They start by expressing Euler's equation in the form

$$\frac{\partial\left(rv_i\right)}{\partial t}=-\frac{\partial\Pi_{ik}}{\partial x_k} \tag{11}$$

where Π_{ik} is the momentum flux density tensor, and is defined as

$$\Pi_{ik}=pd_{ik}+rv_iv_k\,, \tag{12}$$

where d_{ik} is the unit tensor, and Π_{ik} is clearly symmetrical.

Landau and Lifshitz's [4] development is accepted through the expression

$$s'_{ik}=h\left(\frac{\partial v_j}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}d_{ik}\frac{\partial v_l}{\partial x_l}\right)+Vd_{ik}\frac{\partial v_l}{\partial x_l}\,, \tag{13}$$

where s'_{ik} is the viscous stress tensor, and the coefficients of viscosity, h and V, are both positive and independent of velocity.

They then add (not considering the gravitational force per unit mass, Newton's discarded term, the particle random motion term, and the non-gravitational force) the expressions $\partial s'_{ik}/\partial x_k$ to the right side of the Euler equation expressed as equation (14),

$$r\left(\frac{\partial v_i}{\partial t}+v_k\frac{\partial v_i}{\partial x_k}\right)=-\frac{\partial p}{\partial x_i}\,, \tag{14}$$

and obtain

$$r\left(\frac{\partial v_i}{\partial t}+v_k\frac{\partial v_i}{\partial x_k}\right)=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_k}\left\{h\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}-\frac{2}{3}d_{ik}\frac{\partial v_l}{\partial x_l}\right)\right\}+\frac{\partial}{\partial x_i}\left(V\frac{\partial v_l}{\partial x_l}\right). \tag{15}$$

After this they present equation (15) in vector form (which I have slightly rearranged) as

$$-\nabla p + h(\nabla \bullet \nabla)V + \left(V + \frac{1}{3}h\right)\nabla(\nabla \bullet V) = r\left[(V \bullet \nabla)V + \frac{\partial V}{\partial t}\right]. \tag{16}$$

The problem with the foregoing development is that the Euler equations (1) and (14) are both incomplete and in error. As shown previously in this paper, the Euler vector fluid equation (1) needs to be replaced by (in vector form)

$$\left(\frac{-\nabla p}{r}\right) + \left[\frac{2(f-3)p}{3\,r\,r_e}\right]R_e - \left(\frac{M_e G}{r_e^2}\right)R_e - \left(\frac{F}{r}\right) = (v \bullet \nabla)v + \frac{\partial v}{\partial t}. \tag{10}$$

Equation (14), and consequently equations (15) and (16), need similar correction utilizing equation (10). As a side point, when the unit radial vector R_e is collinear with one of the unit vectors for the three dimensions i, k and l, R_e can be replaced by the appropriate unit vector, and r_e can be replaced by the appropriate value of x_i, x_k, or x_l. I continue, though, to use R_e and r_e here in general form.

When the corrections are performed, equation (17) replaces equation (16).

$$-\nabla p + \left[\frac{2(f-3)p}{3r_e}\right]R_e - \left(\frac{r M_e G}{r_e^2}\right)R_e - F + h(\nabla \bullet \nabla)v + \left(V + \frac{1}{3}h\right)\nabla(\nabla \bullet v) = r\left[(v \bullet \nabla)v + \frac{\partial v}{\partial t}\right]. \tag{17}$$

As v approaches the limit zero, and the force F is zero, equation (17) reduces to the herein corrected form of Carpenter's [1] barometric equation (9), confirming equation (17)'s validity and solving the 'meshing with reality' problem presented by the Navier-Stokes equation.
Equation (17) could be further modified to enable the velocity gradients to be large and to allow the general acceleration to vary rapidly, but those developments are beyond the scope of this paper.

3. Discussion

The theoretical discoveries leading to the changes in derivation that yield equations (10) and (17) are of sufficient importance to justify fully the replacement of the Euler fluid and Navier-Stokes equations. If that is not enough justification for some researchers, though, there are experimental consequences too.

In the case of a planetary atmosphere, the fluid densities at upper altitudes {calculated using either of equations (10) and (17) at the limit velocity, v, of zero and with an F of zero} deviate measurably from those calculated using the Newtonian barometric equation. This leads to differences in calculated temperatures and pressures. Similarly, calculation of turbulence parameters for large volumes of atmosphere (such as in typhoons, hurricanes, and even tornados) will be measurably less in error for correct temperatures and pressures than when the present Navier-Stokes equations are used. And anticipations of future conditions in an atmosphere that is chaotic will be reliable for a longer forecast time than when the present Navier-Stokes equations are used.

Previous experiments have not shown any significant deviations from the predictions of the Navier-Stokes equations for several reasons. One of these reasons is that, for Earth's lower atmosphere, the value of 'f' is approximately 5; thus the value of the 'f' term is about +1.33 p/r. This value is opposite in sense to the gravitational term, making it appear to the molecules of the atmosphere that gravity is slightly weaker than it actually is. If the atmosphere were composed of single-atom molecules (such as helium), 'f' would be 3, and the 'f' term would vanish. For a liquid such as Earth's ocean, the value of ' f ' is close to 3, and again the ' f ' term essentially vanishes. These effects are small, and easily overlooked as the cause of some minor problems with applications of the Navier-Stokes equations. Other problems exist with measurements of Earth's atmosphere. The composition of the atmosphere (for example: water vapor, visible moisture, ice particles, carbon dioxide, and even type and amount of dust) varies from packet to packet with altitude and with latitude and longitude. The temperature varies with altitude depending in part on the inconstant solar radiation at various frequencies and, at very high altitudes, with the intensity and altitude shifts of Earth's geomagnetically-trapped corpuscular radiation. These effects cause the measurements to contain enough 'noise' that investigators tend to ignore small deviations.

There is also the fact that only recently have fluids in enclosed spaces (such as the piping on board a spacecraft during launch) undergone sufficient accelerations over long enough periods to reveal problems with the Navier-Stokes equations. The non-gravitational force effects can be quite large and should be readily identifiable. To the best of my knowledge, though, no experiments have yet been performed under such acceleration conditions to quantify this problem. NASA should address this issue if it has not already done so.

The last factor that hides the need for replacement of the Euler fluid and Navier-Stokes equations is that the values for the viscosities h and V are experimentally determined, and the experimental values obtained include automatically but unintentionally at least parts of the correction factors (for some of the terms shown herein to be missing from those equations) under the laboratory conditions of the experiments. The difference here is, of course, the difference between a theoretical prediction based on knowledge (albeit not fully complete) and a prediction using a curve that has been produced by the fitting of the curve to experimental data points.

It is highly doubtful that the replacement vector equations (10) and (17) will be the last of the developments in this series of works to represent more accurately and with more understanding the flow of viscous fluids.

4. Conclusion

For improved fluid equation accuracy, all of the 'static' fluid surrounding a motionless center of gravity must be included in the considerations, as must be the non-gravitational forces (such as those experienced by a constrained fluid in a vehicle undergoing acceleration) and the $v_\perp^2/r$ acceleration effect experienced by single-atom and multiple-atom molecules.

Carpenter's barometric equation must be replaced by

$$\nabla p = -r\, M_{\mathrm{e}} \mathrm{G} R_{\mathrm{e}} / r_{\mathrm{e}}^2 + \left[\frac{2(f-3)p}{3r_{\mathrm{e}}}\right] R_{\mathrm{e}}\,, \tag{9}$$

the Euler fluid equations must be replaced by

$$\left(\frac{-\nabla p}{r}\right) + \left[\frac{2(f-3)p}{3\, r\, r_{\mathrm{e}}}\right] R_{\mathrm{e}} - \left(\frac{M_{\mathrm{e}} \mathrm{G}}{r_{\mathrm{e}}^2}\right) R_{\mathrm{e}} - \left(\frac{F}{r}\right) = (v \bullet \nabla)\, v + \frac{\partial v}{\partial t}\,. \tag{10}$$

and the Navier-Stokes equations must be replaced by

$$-\nabla p + \left[\frac{2(f-3)p}{3r_{\mathrm{e}}}\right] R_{\mathrm{e}} - \left(\frac{r M_{\mathrm{e}} \mathrm{G}}{r_{\mathrm{e}}^2}\right) R_{\mathrm{e}} - F + h(\nabla \bullet \nabla)\, v + \left(V + \frac{1}{3} h\right) \nabla(\nabla \bullet v) = r\left[(v \bullet \nabla)\, v + \frac{\partial v}{\partial t}\right]. \tag{17}$$

References

[1] D.G. Carpenter, A Compensating Term for the 'Side Force Component' Term in the Barometric Equation, *Apeiron* 7(3-4) 159.

[2] D.G. Carpenter , 'An Omission in Sir Isaac Newton's Derivation of the Barometric Equation', *Telicom*, 2000, XIII, No. 1, pp. 76-80.

[3] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, 2nd ed., vol. 6, Translated from the Russian by J.B. Sykes and W.H. Reid, (Butterworth-Heineman, Linacre House, Jordan Hill, Oxford, 1987), pp. 2-3.

[4] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, 2nd ed., vol. 6, Translated from the Russian by J.B. Sykes and W.H. Reid, (Butterworth-Heineman, Linacre House, Jordan Hill, Oxford, 1987), pp. 44-45.

[5] I. Newton, *Principia* (*Mathematical Principles of Natural Philosophy*) *Volume 1.* A. Motte's 1729 AD translation, revised in 1934 by F. Cajori (The University of California Press, Berkeley, 1984), pp. 292-299.