On the Derivation of E = mc^2

Vesselin C. Noninski

vesselin.noninski@verizon.net

Abstract

Analysis is presented of the derivation in [1] of what is popularly known as E = mc^2. It is emphasized that once a relationship, describing a phenomenon in the stationary system, is known exactly and with certainty, any theory that would derive a different relationship regarding the same phenomenon in terms of the same stationary system should be rejected out of hand.

In a series of papers problems in the Special Theory of Relativity (STR) [1] were discussed [2-5]. Probably the most important in this series is the paper "On the Simultaneity of Two Events" which, curiously, does not contain a single formula. In [4] a simple *gedanken* experiment is presented which appears to present a powerful enough argument to pinpoint the most significant, crucial problem in STR – the determination in STR to present the seeming as real.

This paper is dedicated to the analysis of the derivation claimed in [1] of the inertial mass equivalent of energy. The derivation in question, unfortunately, is found to suffer from the consequences of the mentioned crucial problem in STR – the determination to present the seeming as real.

Indeed, on p.61 of [1] it is clearly assumed that in the stationary system K the law of motion of an electron (say, the x-axis component) is

$$m\frac{d^2x}{dt^2} \;=\; \varepsilon X$$

where x denotes the coordinate of the slow moving electron, X is the x-axis component of the electric field, m is the mass of the electron and ε is the charge of the electron.

According to the First Postulate – "The Principle of Relativity" – the above law of motion will have the form

$$m\frac{d^2\xi}{d\tau^2} \;=\; \varepsilon X'$$

in the moving system k (electron's own system), where ξ and τ are the

coordinates of the abscissa and time in the moving system k and X' is the transformed component of the electric field.

So far all is well and good – once the First Postulate is assumed to hold good there could be no objection to writing the law of motion of the electron in k in this way (**cf. Addendum 1**).

Surprisingly, however, the further proposal in [1], namely, to apply the Second Postulate (through the application of Lorentz transformations) to "transform the above equations of motion from system k to system K", that is, to represent the transformed equation $m\dfrac{d^2\xi}{d\tau^2} = \varepsilon X'$, valid in k, in terms of the coordinates of the stationary system K, leads to a paradoxical result. Thus, after application of the Lorentz transformations on $m\dfrac{d^2\xi}{d\tau^2} = \varepsilon X'$ the form of that equation in terms of the coordinates of the stationary system K is claimed to be[*)]:

$$\beta^3 m \frac{d^2x}{dt^2} = \varepsilon X$$

However, as already clearly emphasized above, the relationship between m, $\dfrac{d^2x}{dt^2}$, ε and X can be expressed in no other way but as $m\dfrac{d^2x}{dt^2} = \varepsilon X$. Notice that each one of these components in the latter equation, namely m, $\dfrac{d^2x}{dt^2}$, ε and X, clearly refer to the stationary system K and are not notations for anything else but the mass, acceleration and charge of the electron in the stationary system K in the presence of electric field in the stationary system K. Furthermore, their magnitudes are fixed and are as defined at the outset when we wrote $m\dfrac{d^2x}{dt^2} = \varepsilon X$.

Therefore, any claim that the relationship between m, $\dfrac{d^2x}{dt^2}$, ε and X can be anything else but $m\dfrac{d^2x}{dt^2} = \varepsilon X$ should be rejected out of hand and the theory that would bring about such claim should be discarded as invalid. Thus, a theory whose application leads to a relationship between m, $\dfrac{d^2x}{dt^2}$, ε

and X of, say, the form $\beta^3 m \dfrac{d^2 x}{dt^2} = \varepsilon X$ while considering $\beta^3 \neq 1$ (derived by STR) should be discarded. In addition, obtaining $\beta^3 m \dfrac{d^2 x}{dt^2} = \varepsilon X$ for $\beta^3 \neq 1$ instead of $m \dfrac{d^2 x}{dt^2} = \varepsilon X$ obviously is in a clear violation of the First Postulate – "The Principle of Relativity". Thus, because $\beta^3 m \dfrac{d^2 x}{dt^2} = \varepsilon X$ for $\beta^3 \neq 1$ incorrectly describes the relationship between m, $\dfrac{d^2 x}{dt^2}$, ε and X, deriving, carried out in [1] (**cf. Addendum 2**), on its basis the expression for the energy of motion of the electron W, namely

$$W = \frac{mc^2}{\sqrt{1 - \dfrac{v^2}{c^2}}} - mc^2$$ (popularly known as E = mc^2) is unacceptable.

If, however, in an attempt to save STR, one accepts that $\beta^3 = 1$ then the desired relativistic expression for the energy of motion of the electron

$$W = \frac{mc^2}{\sqrt{1 - \dfrac{v^2}{c^2}}} - mc^2$$ cannot be derived (**cf. Addendum 2**).

So, here we are faced with the same problem as in the case with the simultaneity of two events, discussed in [4] – results of derivations based on two postulates lead to a relationship in obvious contradiction with the truthful relationship, established as such at the onset.

One can play devil's advocate and object that in the transformed equation $\dfrac{d^2 x}{dt^2} = \dfrac{\varepsilon}{\beta^3 m} X$ the form of the second derivative $\dfrac{d^2 x}{dt^2}$ is the way an observer in the moving frame k is seeing it and not the way an observer in the stationary frame K sees it. But then, it should be definitively pointed out that the observer in the moving frame k "sees" $\dfrac{d^2 x}{dt^2}$ as $\dfrac{\varepsilon}{\beta^3 m} X$ and not, correctly, as $\dfrac{\varepsilon}{m} X$ only due to STR and this means that the said observer deceives himself by using STR because the form of $\dfrac{d^2 x}{dt^2}$ obtained by STR is incorrect. Any theory that claims to have physical meaning requires the

observer in the moving system to tell truthfully what the expression $\dfrac{d^2x}{dt^2}$ actually looks like in the laboratory frame K and not what it just appears to him to be. Ships on the horizon appear as specs for an observer on the shore but no reasonable observer would ever insist that that is the actual size of these ships, just because they appear so. In the discussed case (as with the ships) we actually know from the outset exactly what the expression $\dfrac{d^2x}{dt^2}$ looks like in the laboratory frame K, namely $\dfrac{d^2x}{dt^2} = \dfrac{\varepsilon}{m}X$, which obviously differs from $\dfrac{d^2x}{dt^2} = \dfrac{\varepsilon}{\beta^3 m}X$ for $\beta^3 \neq 1$.

The above assertion, namely that $\dfrac{d^2x}{dt^2} = \dfrac{\varepsilon}{\beta^3 m}X$, is an acceptable rendition, from the point of view of the moving system, of what in the stationary system is clearly established to be $\dfrac{d^2x}{dt^2} = \dfrac{\varepsilon}{m}X$ is as curious and untrue as, say, an assertion that it is acceptable from the point of view of a moving system 2 + 2 to appear as 5 in the stationary system. Because it is well established beyond any doubt that in the stationary system 2 + 2 = 4, any theory that will bring about a result different from 4 when adding 2 and 2 in the stationary system should be rejected out of hand. Thus, any theory which will claim an equation different from $\dfrac{d^2x}{dt^2} = \dfrac{\varepsilon}{m}X$ in the stationary frame will have to be abandoned out of hand.

Again, however, the devil's advocate may object: "well, this expression $\beta^3 m \dfrac{d^2x}{dt^2} = \varepsilon X$ has nothing to do with the stationary system. This is in fact the transformed equation just presented in terms of the coordinates of the stationary system. This is just showing what the transformed equation looks like if we choose to represent it in terms of the stationary coordinates. It is not true that it does not obey the First Postulate because the First Postulate causes the coordinates in the moving system to be primed. If we represent it in terms of the un-primed coordinates then the transformed equation may have any form."

However, in writing not just anything else but these concrete

parameters m, $\dfrac{d^2x}{dt^2}$, ε and X one in no uncertain terms expresses exactly what the meaning of these parameters is. These parameters are not a substitute for anything else or are some arbitrarily chosen letters but are the mass, acceleration and the charge of the electron as well as the electric field in the stationary system.

This argument can be seen clearly also by considering the straightforward methodology used in [1] when transforming Maxwell's Equations on p.52. Let us recall what that methodology is: The author first applies the First Postulate – "The Principle of Relativity" – by rewriting Maxwell's Equations in the moving system in the same form as their form in the stationary system, only replacing the coordinates and the fields with those of the moving system. Then, as a second step, he starts again with the Maxwell Equations in the stationary system but this time carries out Lorentz transformations on them to obtain the form these equations will acquire in the moving system k. And in the end, declaring the obvious, namely, that the two ways of transforming of Maxwell's Equations – on the one hand by applying the First Postulate and on the other hand by applying the Second Postulate (Lorentz transformations) – should give the same thing, he compares the two sets of equations and derives what he claims to be the Lorentz force (for the record, in [2] it is shown that in fact Lorentz force has not been derived in [1] despite the claim therein).

Let us apply the same methodology in the case we are discussing here. First, apply the First Postulate on $m\dfrac{d^2x}{dt^2} \;=\; \varepsilon X$ to obtain

$$m\frac{d^2\xi}{d\tau^2} \;=\; \varepsilon X'$$

Then, as a separate exercise, apply the Second Postulate (Lorentz transformations) on $m\dfrac{d^2\xi}{d\tau^2} \;=\; \varepsilon X'$ to obtain[**)]

$$\beta^3 m\frac{d^2x}{dt^2} \;=\; \varepsilon X$$

These two equations pertaining to the stationary system K, namely

$$m\frac{d^2x}{dt^2} = \varepsilon X \text{ and } \beta^3 m\frac{d^2x}{dt^2} = \varepsilon X \text{, "must express exactly the same}$$
thing." Therefore, it follows that

$$\beta^3 = 1$$

since in the initial equation $m\dfrac{d^2x}{dt^2} = \varepsilon X$ there is no additional function of

velocity v. However, if $\beta^3 = 1$, then the derivation shown on p.63 leading

to $W = \dfrac{mc^2}{\sqrt{1-\dfrac{v^2}{c^2}}} - mc^2$ cannot occur (**cf. Addendum 2**).

    The above considerations clearly reveal that the relationship

$W = \dfrac{mc^2}{\sqrt{1-\dfrac{v^2}{c^2}}} - mc^2$ claimed in [1] cannot be derived and demonstrates

the non-physical nature of STR.

Discussion

This analysis is another illustration in addition to previously published arguments [2-6] of the fact that the claims by STR advocates that it is devoid of internal inconsistencies and that its falsifying can only occur through experimental testing are unfounded. No experimental testing is necessary to verify that a theory leading to an outcome that $2 + 2 = 5$ or that Newton's

Second Law of Motion on Earth is $F = \dfrac{m}{f(v)} a$ is an incorrect theory.[***)]

    Nevertheless, as a separate exercise, it may be interesting to understand how the claimed numerous experimental confirmations of STR could be explained away. It has already been pointed out that claimed significant experimental confirmations (e.g. experiments with μ-mesons, Michelson-Morley experiment) which have been considered as crucial proof for the validity of STR are actually of questionable value as confirmations of STR [4,6].

    Often the very existence of accelerators of charged particles is presented as a proof for the experimental confirmation of STR [7,8].

Advocates claim it would not have been possible to even build these accelerators and have them properly working if relativistic corrections are not applied in their construction. One can even hear them state that they see confirmation of STR every day, although it is true that one can also deceive oneself every day. In the standard literature on particle accelerators [7,8] there are sections dedicated to "relativistic particles", i.e. particles for which relativistic effects are claimed to be of demonstrable magnitude. Even accelerators of special constructions have been designed such as synchrocyclotrons whose construction is said to have been specifically based to account for such effects. The mentioned accelerators are large facilities involving complicated tasks to solve when building them which require substantial engineering ingenuity. One may speculate that accounting for the relativistic effects when such sizable engineering projects are conducted may be one of many concerns of comparable and even higher priority. It may be supposed that sheer historical tradition may have played a role in presenting the relativistic corrections in the standard texts as having the claimed substantial role. These are, of course, all speculations and further clarification as to why experiments with accelerators appear to be confirming STR is still outstanding and will be the subject of future communications.

  Probably, one may speculate even at this point that if the equation $E = mc^2$ happens to be confirmed as an empirical dependence it could be a demonstration of the peculiar properties of light and especially the peculiar dependence $E = pc$, where p is linear momentum, assumed for photons [11,12] and has nothing to do with STR.

  One significant problem that stands in the way of a pursuit to clarify, one way or another, the experimental aspect of the raised questions with STR is that access to accelerators is not easy, not to speak of the possibility to independently verify the relativistic claims through direct carrying out of experiments there. Thus, one has to rely only on second hand reports, usually in the popular literature, that such confirmations of the relativistic effects in accelerators are available.

  As far as the scientific literature on the subject is concerned, this author is of the impression (maybe wrong !) that studies specifically dedicated to verification of the necessity for relativistic corrections in the high-energy particle field are not abundant. It appears to this author that the culture and the tradition in this field of research requires the perception that relativity is firmly established and there are much more important problems to solve with the available accelerators (or use them for practical, say,

medical purposes) than to dedicate resources to insignificant and useless tasks of verifying STR.

One example of a publication specifically dedicated to study STR effects in accelerators is [9]. However, as Petrov [10] has pointed out to me, even a superficial inspection of the widely cited paper [9] reveals that the five experimental points presented therein, used to show agreement with the predictions of STR, have not been acquired at comparable conditions.

Could it be, in view of the above, that the large body of work carried out on STR and its applications has in fact dealt with illusory perceptions which do not conform with the real phenomena ? What is then the impression created in society of the great successes of STR due to ? Such questions probably have more or less a sociological aspect since they concern the way society perceives science, rather than having available rigorous scientific answers. Of course, if one wishes one can build beautiful and probably amusing constructs on such illusory perceptions – after all poetry and literary fiction are examples of such legitimate and necessary human pursuits. At that transcending the limits of science into fiction is especially to the liking of some popular writers and some audiences and this explains the tangible interest in the subject discussed – for some entertainment is of more value than the truth about a phenomenon and this is what sells. This author is of the opinion, however, that although there is a place in life for fiction, fantasies and fun, as far as science is concerned truth about physical phenomena is the goal and this must not be substituted by fictitious constructs presenting illusions as reality.

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[*] Curiously, when carrying out Lorentz transformations on Maxwell's Equations on p.52 [1], from which the expressions of the Lorentz force on p.54 of [1] emerge, on same p.54 the author requires neglecting of "the terms multiplied by the second and higher powers of $\dfrac{v}{c}$" so that the known form of Lorentz force is obtained (as shown in [2] this, however, is not achieved to any degree of approximation). Now, in the derivation discussed here (p.62 of [1]) the author prefers to forgo such analysis.

[**] The form of the transformed equation in K is as shown on p.62 of [1]; one may wonder how the cubic power, β^3, appeared, exactly fitting the requirement for the final derivation (any other form will not do).

[***] Recall that STR was created only with a "pencil and a paper". Why, if there are problems with it, should it not be possible to reveal these problems with a "pencil and a paper" too ?

Acknowledgements

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Addendum 1

Note that whatever conditions (say, $v << c$) the equation $\dfrac{d^2x}{dt^2} = \dfrac{\varepsilon}{m}X$ is derived for in K, these

same conditions apply when transforming it into the moving system k $m\dfrac{d^2\xi}{d\tau^2} = \varepsilon X'$.

Addendum 2

We imagine there is a magnetic field in the laboratory frame K with components X, Y and Z. Suppose an electron of charge ε is stationary versus K. Then, the Coulomb force (say its x-axis component) which the electron will feel will be $F = \varepsilon X$. Now, according to Newton's law this force has to be equal to the mass of the electron times its acceleration. We already said that the electron is immovable versus K so it appears that the acceleration of the electron is zero. We may, however, imagine a next, very close instant of time and we can write

$$F = m\frac{d^2x}{dt^2} = \varepsilon X$$

We may carry out the same observation for the electron in its own frame k. In frame k the electron is also stationary versus k. However, we will think of some very close next moment and we may write

$$F' = m\frac{d^2\xi}{d\tau^2} = \varepsilon X'$$

The above considerations are made approximately as in [1]

To derive the desired equation first a transformation from the moving frame k into the stationary

frame K is made of $F' = m\dfrac{d^2\xi}{d\tau^2} = \varepsilon X'$ and as far as X is concerned the result presented on

p.62 of [1] is

$$\varepsilon X = m\beta^3 \frac{d^2x}{dt^2}$$

This is then substituted on p.63 of [1] in the formula for the energy of motion of the electron

$W = \int F dx$. Recall that the component of electric force along the x-axis, according to Coulomb's law, is $F = \varepsilon X$ where ε is the electric charge (of the electron in this case) and X is the x-component of the electric field. Thus, when substituting we have:

$$W = \int F dx = \int m\beta^3 \frac{d^2 x}{dt^2} dx$$

$$= m \int \beta^3 \frac{dv}{dt} dx$$

where $\dfrac{dx}{dt} = v$ is the velocity. Thus, we have

$$= m \int \beta^3 v \, dv$$

or

$$= m \int \frac{v}{\left(\sqrt{1 - \frac{v^2}{c^2}}\right)^3} dv = m \frac{1}{2} \int \frac{1}{\left(\sqrt{1 - \frac{v^2}{c^2}}\right)^3} dv^2$$

and multiplying and dividing by c^2 we get

$$= m \frac{1}{2} \int \frac{c^2}{\left(\sqrt{1 - \frac{v^2}{c^2}}\right)^3} d\frac{v^2}{c^2} = mc^2 \frac{1}{2} \int \frac{1}{\left(\sqrt{1 - \frac{v^2}{c^2}}\right)^3} d\frac{v^2}{c^2}$$

We can also represent this as

$$= - mc^2 \frac{1}{2} \int \frac{1}{\left(\sqrt{1 - \frac{v^2}{c^2}}\right)^3} d\left(1 - \frac{v^2}{c^2}\right) = - mc^2 \frac{1}{2} \int \left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}} d\left(1 - \frac{v^2}{c^2}\right)$$

which is

$$= - mc^2 \frac{1}{2} \int (-2) \, d\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = mc^2 \int d\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Once we have the form of the indefinite integral we may now integrate from $v = 0$ to $v = v$

$$= mc^2 \int_0^v d\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

or

$$E_k = \frac{m_e c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_e c^2$$