Noncommutation

Abstract

It is shown that the reason quantum variables do not commute is because the initial state of a quantum system is indeterminate.

1.0 Introduction

1.1 Commutation

The physical variables of a classical oscillator such as a vibrating string refer to point particles on one-dimensional trajectories so their order of multiplication as canonical conjugates makes no difference. In other words, we say that classical variables commute. This may be expressed mathematically as pq - qp = 0, where p is the momentum and q is the position.

1.2 Non-commutation

In 1925 Heisenberg was surprised to find that quantum mechanical variables do not commute[1]. He expressed non-commutation formally as follows:

$$\mathbf{pq} - \mathbf{qp} = \hbar \qquad 1)$$

where **p** and **q** are not numbers; but rather arrays of quantities, or matrices. The diagonal elements are steady states while the off-diagonals are either absorptions or emissions. The complete matrix has an infinite number of components and corresponds in its entirety to one of the dynamic variables of Newtonian theory. *Whereas classical variables are defined by a single number each component of the matrix is associated with any two of an infinite number of orbits.*

2.0 Quantum mechanical systems

2.1 Energy quantization

A physical system commonly used to illustrate non-commutation, and also the one used by Heisenberg, consists of a hydrogen atom stimulated by an electromagnetic wave. If the electron is raised into a higher orbital so that the fields are quantized a photon is irreversibly emitted[2]. This may be shown schematically by using an energy diagram (figure 1), where 1 and 2 denote energy levels and arrows refer to transitions. On the left the energy of an electron increases and then decreases, while on the right the reverse occurs. In each case a photon is emitted. The photon's energy is given by the energy difference between the two energy levels. A single energy level, an electron's energy state, has no physical significance since it cannot be detected. Energy states can only be detected in pairs by means of energy differences. Detection is possible when a photon is emitted so that emission refers to both energy states[3]. The two processes are identical when described in terms of energy differences.

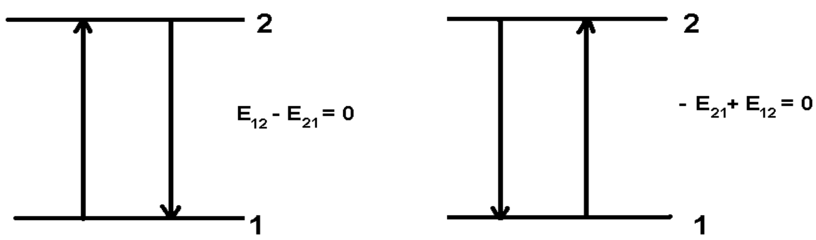

Figure 1

2.2 Angular momentum of an atomic system

Now consider what happens when the same two energy exchanges are analyzed in terms of the momentum. Using Compton's equation for the momentum of a photon, p=h/λ, the first exchange may be expressed:

$$p_{12}\lambda_{12} - p_{21}\lambda_{21} = 0 \qquad 2)$$

Angular momentum increases by an amount ђ when the electron is excited and is then reduced by the same amount when the atom returns to its ground state. Thus this type of photon emission ends up with the atomic system in its ground state.

The energy exchange on the right of figure 1 is described by the following expression:

$$p_{21}\lambda_{21} - p_{12}\lambda_{12} = \hbar \qquad 3)$$

The electron begins in an excited state, reverts to the ground state by emitting a photon, and is excited once again. Thus the final state of the atomic system has an angular momentum that is greater than the ground state by an amount ђ. In both cases 2) and 3) a photon is emitted, but because the order of the physical variables changed the angular momentum of the atomic system described by 3) is greater than 2). Thus the physical variables do not commute. It is not possible to avoid non-commutation by anticipating the order of physical variables because the state of a quantum system is unobservable. We conclude therefore that *quantum mechanical variables do not commute because the initial state of a quantum system is indeterminate.*

3.0 Conclusion

Equations 2) and 3) may also be used to interpret the Heisenberg energy matrix described in 1.2. Equation 2) represents the diagonal elements of the matrix which correspond to "steady states" in the Heisenberg interpretation. They describe the change of angular momentum that occurs when an electron leaves and then returns to the same orbital. Equation 3) represents the off-diagonal elements of the matrix.

1 An historical account is found in J. Mehra & H. Rechenberg, *op. cit.*, p. 226. M. Jammer, *The Conceptual Development of Quantum Mechanics* 2nd ed. (NY: Tomash, 1989), p. 213.
2 For purposes of discussion only two orbitals are considered.
3 Because there is no such thing as zero energy absolute energy values do not exist.