ORIGIN AND DEVELOPMENT OF THE THEORY OF RELATIVITY

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Newton and Galileo

There is an explicit statement of a restricted form of what Einstein later called the Principle of Relativity in Sir Isaac Newton's *Principia* [1]. It may be translated:

"The motions of bodies enclosed in a given space are the same relatively to each other whether that space is at rest or moving uniformly in direction [*i.e.* moving with a constant velocity in a straight line] without circular motion."

The sense of the term "space" in the foregoing passage is what we have in mind at the present time when we speak of an "inertial system of reference." This restricted principle of relativity is also implicit in the work of Galileo and indeed Einstein has called it the *Galilean Principle of Relativity.* The best description of it is: "the equations of mechanics have the same form in all inertial co-ordinate systems." An inertial system is appropriately defined as one in which Newton's first law of motion holds.

Newton's eighteenth-century successors, Maclaurin, Euler, d'Alembert, Lagrange and others, made great contributions to the form and application of his mechanics; but their work is not distinguished by anticipations of later developments of physical theory and we shall not be concerned with it. We except that of Moreau de Maupertuis of course, since his *Principle of Least Action* anticipates in some degree and is intimately related to the more profound and comprehensive statement of Sir William Rowan Hamilton, Royal Astronomer in Ireland in the earlier half and middle of last century. Carl Neumann, in his inaugural lecture (November 3, 1869) on appointment as professor of mathematics in the old University of Leipsic [2] and [3], gave a clear statement of the Newtonian principle of relativity and his Body Alpha is in some degree suggestive of later advances.

Hamilton

Hamilton's form of Newtonian mechanics was developed during the period extending from about 1830 to 1835. His most remarkable discovery (which apparently made no impression on his contemporaries, not even on Jacobi it seems, and appears to have been first appreciated by Louis de Broglie and Erwin Schroedinger in the development of wave mechanics) was that the laws of geometrical optics and those of Newtonian mechanics are identical in form [4]. The principle of least action of de Maupertuis, which, as we have just said, anticipates in some degree Hamilton's still more general principle, has *exactly* the form of Fermat's principle, provided we express this latter in terms of the undulatory theory of light. Fermat's principle gives a stationary value to the integral $\int_A^B \nu' dq$, where ν' means the wave number (number of waves in the unit length) so that the integral expresses the number of waves between the points A and B on a ray of light. Now de Maupertuis' principle assigns a stationary value to the integral $\int_A^B p dq$, where p means the momentum of a particle and the integral is extended over the path of the particle from A to B. Clearly the momentum p, and the wave number, ν', are analogous to one another. There is still another point: the variation of the Fermat integral is subject to the constancy of the frequency, ν, while the de Maupertuis variation is subject to the constancy of the energy, H; so that ν and H are analogous.

Hamilton's wider principle assigns a stationary value to $\int_\alpha^\beta (p dq - H dt)$, if for simplicity we confine our attention to a single particle.* The terminal spatial points of the actual path of the particle and of the varied path coincide. They also begin at the same time and end simultaneously. It is preferable to write the integral in the form:

$$\int_\alpha^\beta (p_x dx + p_y dy + p_z dz - H dt) \qquad . \qquad . \qquad (1)$$

This expression obviously anticipates Minkowski's space-time continuum and will be discussed in some detail later.

* It is important to bear in mind that Fermat's principle can be expanded to take the form of giving a stationary value to $\int_\alpha^\beta (\nu' dq - \nu dt)$. This gives the equations of geometrical optics exactly the form of Hamilton's canonical equations [4].

In geometrical optics we are concerned with very short waves, with waves so short indeed that diffraction patterns are too minute to be observable, and Maxwell's theory indicates [5] that in the case of such very short waves

$$c^2 = uv \quad . \quad . \quad . \quad . \quad (2)$$

where c is the velocity of light in empty space when referred to an inertial co-ordinate system—the experiments of Michelson and Morley revealed c to be a constant, independent of the particular inertial co-ordinate system relatively to which it is measured [6] —and u and v are respectively the phase velocity and the group velocity. Now the phase velocity is obviously ν/ν'—it is identical with the quotient of wave-length by period of vibration and the Hamiltonian analogy described above gives us

$$u = \frac{\nu}{\nu'} = \frac{H}{p}.$$

Therefore, by (2)

$$\frac{H}{p} = \frac{c^2}{v}.$$

Consequently

$$\frac{\text{energy}}{\text{momentum}} = \frac{mc^2}{mv},$$

i.e.

$$\boxed{\text{energy} = \text{mass} \times c^2} \quad . \quad . \quad . \quad (3)$$

This is Einstein's famous formula expressing the identity of mass and energy.

We may therefore write the expression which appears under the symbol of integration in (1) in the form

$$p_x dx + p_y dy + p_z dz - mc^2 dt.$$

This suggests the form

$$p_x dx + p_y dy + p_z dz + p_w dw \quad . \quad . \quad . \quad (4)$$

and remembering that $p_x = m\frac{dx}{dt}$ we write $p_w = m\frac{dw}{dt}$ so that

$$m\frac{dw}{dt}dw = - mc^2 dt.$$

It follows at once that

$$dw = \sqrt{-1}\, c dt,$$

and

$$p_w = m\sqrt{-1}\, c,$$

in agreement with Minkowski's form of the later special relativity. Obviously it was the form of the analogy between geometrical optics and mechanics which caused Hamilton to express the latter by the use of momenta, rather than velocities, as did Lagrange.

Clerk Maxwell

The great advance made by J. Clerk Maxwell was suggested, as is well known, by Faraday's view that the di-electric medium in which, *e.g.* an electric field has been established, is in a state of strain. What is perhaps not so well known is that Maxwell, in identifying the electric field intensity with the stress, was led to see that the strain consisted in a displacement of electricity. The energy per unit volume is $\frac{1}{2}$ stress $\times$ strain and when this is equated to $KE^2/8\pi$, where K is the di-electric constant and E the electric field intensity, the strain turns out to be equal to $KE/4\pi$. Imagine now a particle with a charge Q placed at the centre of a spherical surface of radius r and substitute Q/Kr^2 for E in this expression for the strain. The result is

$$\text{strain} = \frac{Q}{4\pi r^2}.$$

We must identify it with quantity of electricity displaced through the unit area of the spherical surface.

Voight, Lorentz and FitzGerald

The result of the experiments of Michelson and Morley was very puzzling to the physicists of their time, for reasons which are well known. Both FitzGerald and Lorentz, independently of one another, enunciated the famous contraction hypothesis to account for this result. They thought of it as a real contraction of materials in consequence of their motion through the aether, whereas, as we shall see presently, it is just of the same nature as the change in, *e.g.* the X co-ordinate of a point when we pass from one system of co-ordinates to another. To believe in the actuality of the contraction and also in the reality of the aether was very natural in the time of Lorentz; but made it difficult to advance further. Nevertheless he did reach, almost correctly, the famous transformation, now named after him, which appears to have been expressed correctly for the first time by Einstein, though much earlier, *c.* 1887, W. Voight reached a transformation [7] which only differs in a trivial way from that of Einstein.

Einstein and Minkowski

In Einstein's original paper [8] there is a passage which may be translated as follows:

". . . in all those co-ordinate systems in which the mechanical equations hold, the same electromagnetic and optical equations hold also, as has been proved already for first order quantities. We will raise this supposition (the content of which we shall call the 'Principle of Relativity' in what follows) to the rank of a premiss together with the supposition, which is only apparently incompatible with it, that light travels in empty space with a definite velocity, V [nowadays we use the letter c for this] which is independent of the state of motion of the emitting body."

This passage means of course that the equations of electromagnetism and optics have the same form in all inertial co-ordinate systems. It is in fact an extension of the old Newtonian principle of relativity to apply not only to mechanics, but also to electromagnetism and optics. From these premisses Einstein reached what we believe to be the precisely correct form of the Lorentz transformation, though there are many indications in this earliest paper that he did not, at that time, grasp its full implications. Had he done so he might have anticipated Minkowski's great discovery of the space-time continuum [9]. Indeed the adoption of the Michelson-Morley result should have revealed to him the invariance of

$$dx^2 + dy^2 + dz^2 - c^2dt^2$$

in empty space when referred to an inertial co-ordinate system and the co-variance of electromagnetic equations when referred to such co-ordinate systems. But this was first recognised by H. Minkowski a year or two after Einstein published his great paper. Minkowski's form of special relativity completely elucidates the nature of the "contraction" assumed by FitzGerald and Lorentz. The length of a body is now comparable to a distance measured in the direction of a co-ordinate axis and will vary when we pass from one co-ordinate system to another though no actual change occurs in the body. There is of course a corresponding change in the numerical specification of a period of time—for instance of the length of the life of a human being.

Present-day Descriptions of Einsteinian Relativity

It is of great interest to study the various descriptions of Einsteinian relativity, especially those written in recent times. We shall begin with Bertrand Russell's *ABC of Relativity* which, being one of the earlier works on the subject, may be forgiven for at least some of its weaknesses. One of the most striking things in the works of this author is the way in which he uses words. For example, we read on page 12: "As a matter of fact two billiard balls never touch at all." Comment seems quite unnecessary. Certainly their momenta undergo sudden changes now and again! This happens when the distance between their centres is equal to the sum of their radii. This is what we mean when we say that they are touching. What does Bertrand Russell mean when he uses the word "touch"?

The physicist gives the name "*force*" to rate of change of momentum; but we read on the same page of Russell's book that before Einstein's theory of relativity "'Force' was known to be merely a mathematical fiction." What shall we say about this kind of nonsense? We cannot do better than borrow the words of the French mathematician, Lebesgue, "*Sans doute, on pourrait chicaner sur le mot* 'force'." *

On pages 21–2 of Russell's book we read: ". . . if you take a cube and move it very fast, it gets shorter in the direction of its motion, from the point of view of a person who is not moving with it though from its own point of view (*i.e.* for an observer travelling with it) it remains just as it was." This is the type of error that is perhaps most frequently made in the most recent accounts of Einstein's theory. A body moving in a straight line with a constant velocity referred to an inertial co-ordinate system does not change in any way—so long as it is not interfered with. It does not vary with "the point of view of a person who is not moving with it. . . ." What its velocity and dimensions may be is determined by the *co-ordinate system* to which it is referred!

Professor McCrea writes in his *Relativity Physics*: † "The special theory of relativity deals with observers in uniform relative motion." Of course it does nothing of the kind. It is likely that this kind of misunderstanding is partly due to E. A. Milne's Kinematic Relativity, though Milne himself was under no misapprehension about Einsteinian theory. He writes, quite correctly: "The important thing [in Milne's theory] is not transformations of co-ordinates [as in Einstein's theory] but transformations of observers, from one observer to another observer." ‡

It is convenient to stop at this point since an account of the

* H. Lebesgue: *Leçons sur l'Intégration*, 2nd Ed., p. 335.

† Methuen's Monographs.

‡ *Relativity, Gravitation and World Structure*, p. 5.

advance to the General Theory of Relativity must occupy much space. It may be said, however, that Einstein's gravitational theory is also indicated in the earlier stages of the development of physical theory.

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