The Sagnac Effect, Once More

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A simple and rigorous proof of the Sagnac effect for the most general value of the synchronization parameter e_1 is given. If in the final result one adopts the relativistic e_1 one can see that the relativistic theory is incompatible with the experimental evidence. Only the theory with $e_1 = 0$ predicts correctly the Sagnac effect. Of course the obtained results depend on the assumptions made, which look rather safe however.

1. Introduction

In the 1913 Sagnac experiment [1] a platform rotated uniformly at a rate of 1-2 rot/sec. In an interferometer mounted on the platform, two interfering light beams, reflected by mirrors, propagated in opposite directions along a closed horizontal circuit. The rotating system included also the luminous source and a photographic plate recording the interference fringes. Sagnac observed a shift of the interference fringes every time rotation was modified. This shift depends on the relative time delay Δt, object of our calculations, with which the two light beams (better: localized light pulses) reach the detector.

Authoritative attempts at explaining the Sagnac observations in terms of the relativistic theories were made by Langevin [2], Post [3] and Landau and Lifschitz [4], but they all found it impossible to carry through a purely deductive argument and added arbitrary additional assumptions with a big loss of generality. Almost a century after the 1913 discovery of the Sagnac effect no justification of it exists based on special and/or general relativity. Hasselbach and Nicklaus (1993) list about 20 different "explanations" of the effect and comment:

> "This great variety (if not disparity) in the derivation of the Sagnac phase shift constitutes one of the several controversies ... that have been surrounding the Sagnac effect since the earliest days of studying interferences in rotating frames of reference." [5]

In recent years the Sagnac effect finally received a complete explanation from a theory based on absolute simultaneity, weak relativity [6]. At the same time it became clear why the TSR is unable to explain the effect, but very few people accept these conclusions. At least partly there is a problem of clarity of the quoted paper and the present lines are directed at making the arguments of [6] as clear as possible.

2. The First Six Assumptions

1. An isotropic inertial frame S_0 exists such that relative to S_0 the velocity of light is "c" in all directions.
   $\Rightarrow$ In S_0 clocks are synchronized with the Einstein method;
   $\Rightarrow$ one way velocities relative to S_0 can be measured.
2. Space is homogeneous-isotropic and time homogeneous, at least for observers at rest in S_0.
3. Considering a second inertial frame S we assume that the origin of S, observed from S_0, moves according to the equations $x_0 = Vt_0$, $y_0 = z_0 = 0$.
4. The Cartesian axes of S and S_0 coincide for $t = t_0 = 0$.

We add two assumptions based on solid empirical evidence [6]:

5. The two way velocity of light is the same in all directions and in all inertial frames.
$$c_2(\theta) = c \tag{1}$$
6. Clock retardation takes place with the usual factor R if calculated with respect to S_0:
$$R = \sqrt{1 - V^2/c^2} \ . \tag{2}$$

It should be stressed that the TSR satisfies all these assumptions. Usually one takes for granted the validity of the first four. The two famous postulates of the TSR (relativity principle and invariance of light velocity) are here replaced by the weaker assumptions 5 and 6.

3. The Seventh Assumption

A further assumption, often left implicit, is the so called "acceleration hypothesis" (AH). We use the AH for a rotating platform, but its extension to any type of acceleration frame is straightforward. It works in two ways, either one applies to the accelerated frame the physical properties of a corresponding inertial frame, or one extends to the inertial frame the physics developed locally in the accelerated frame. It is well known that Einstein developed the general theory of relativity by introducing an active role of acceleration via a conjectural "gravitational" potential of the fictitious forces [7]. Thus he contradicted the AH, which he had used in 1905 [8] in the case of the clock paradox. More recent results, especially those by Builder [9], Prokhovnik [10], Selleri [11] and Unnikrishnan [12], have however led to the conclusion that the conjectural gravitational potential of the fictitious forces cannot have any effect on the time marked by the clocks.

An assumption concerning the propagation of light relative to accelerated frames is clearly needed since a priori we have no information about this side of relativistic physics. The empirical evidence shows that accelerations do not produce physical phenomena. For example, the CERN muon storage ring experiment [13] shows that an acceleration as large as 10^{18} g does not modify appreciably the muon lifetime. In the same experiment was instead very visible the effect of velocity, with an increase of the muon lifetime from τ to τ_0 by a factor of about 28 according to

$$\tau_0 = \tau / R \ , \tag{3}$$

where R is given by (2). This growth is exactly equal to the lifetime increase of a linear beam of muons having the speed v. It is evident that accelerations are no protagonists of the game of physics. One could half jokingly say that accelerations do not exist, as they are only velocity variations.

Very different is the case of velocities, which are easily recognized as cause of important phenomena. In every small region σ of the rotating platform one can imagine an associated accelerated frame A_σ in which σ is constantly at rest, and a co-moving inertial frame I_σ in which σ is instantaneously at rest such that a statement about physics correct in σ, thought as a part of A_σ must be correct also in I_σ and vice versa. This is the AH.

Velocity is space divided by time. If the length of a rod and the rate of a clock are both not modified by acceleration, the AH must be correct when applied to a speed, such as the speed of light. As far as the present author knows, a dependence of length on acceleration has never been proposed, so that, adding Einstein's 1905 considerations on clock rate invariance under changes of acceleration, the invariance of velocities under changes of acceleration seems a safe assumption. Therefore let us consider a concrete application of the AH to the velocity of light. According to the theory of relativity in an inertial system I_σ the velocity of light is c in all directions. Therefore, using the AH, the velocity of light has to be c in all directions independently of disk rotation, also relative to A_σ. In this way, however, the two light pulses moving in opposite directions along the disk border cross a long set of σ regions, all at speed c, and need the same time to complete the tour, so that the Sagnac effect goes to zero, contrary to empirical evidence.

Having excluded the existence of direct consequences of accelerations, we can extend the negative conclusions to indirect consequences. In particular the resolution of the clock paradox offered by the TGR is invalid. There is no gravitational potential of the fictitious forces. As is well known, the general relativistic effect hypothesized by the TGR is not on the accelerating system, but on the other systems participating in the ideal experiment with a static role. If this idea were correct there should be also an effect of the acceleration of the storage ring muons on other low energy muons eventually present in the laboratory. But nothing of the type has ever been seen and the very proposal of such an idea idea makes it sound very unlikely.

4. The Equivalent Transformations

The first six assumptions determine the transformations of the space and time variables from S_0 to S to have the form of the "Equivalent Transformations" (ET) [6]

$$x = (x_0 - vt_0)/R \quad y = y_0 \quad z = z_0 \quad t = Rt_0 + e_1(x_0 - vt_0) \tag{4}$$

Reichenbach and Jammer believed that the parameter e_1 is free and can be fixed conventionally by synchronizing clocks in S. Sometimes e_1 is called "synchronization parameter". We will see, however, that far from being free e_1 must be zero.

The one way velocity of light consequence of the ET is then:

$$\frac{1}{c_1(\theta)} = \frac{1 + \Gamma\cos\theta}{c} \tag{5}$$

where θ is the angle between light propagation direction in S and absolute velocity of S. The parameter Γ is given by:

$$\Gamma = \frac{V}{c} + c\, e_1 R \tag{6}$$

Of course in the TSR one must have

$$c_1(\theta) = c \quad \Longrightarrow \quad \Gamma = 0 \quad \Longrightarrow \quad e_1 = -\frac{V}{c^2 R} \tag{7}$$

The Eqs. (4) represent the set of theories "equivalent" to the TSR: if e_1 is varied different theories are obtained. According to the Reichenbach-Jammer conjecture they should be equivalent for the explanation of experimental results.

Most textbooks deduce the Sagnac formula in the laboratory, but say nothing about an observer on the rotating platform. Special relativity is self-contradictory, as it predicts a null effect on the platform, but a nonzero value if the platform rotation is studied from the laboratory. Many other theories predict a wrong result. Only the theory with $e_1 = 0$ gives the right answer.

5. Laboratory / Disk Connections

Consider a clock, marking time t, fixed in a point of the moving inertial system S. Seen from S_0 it satisfies the equation

$$x_0 = vt_0 + \bar{x}_0 \,, \tag{8}$$

where $\bar{x}_0$ gives the clock initial position. Substituting x_0 into Eq. (4) we get

$$x = \bar{x}_0/R \tag{9}$$

(giving the fixed x of the clock in S) and

$$t = R\,t_0 + e_1\bar{x}_0 \,. \tag{10}$$

Consider two events taking place at different times in the same point of S. Clearly we must write the previous equation twice, the first with t_1 and t_{01}, the second with t_2 and t_{02}: 1 and 2 distinguish the two events. By subtracting these two equations side by side and defining $\Delta t = t_2 - t_1$ and $\Delta t_0 = t_{02} - t_{01}$ we get

$$\Delta t = R\,\Delta t_0 \,. \tag{11}$$

Naturally, Eq. (11) is predicted by all ET theories, including the TSR, in case of uniform motion, as it is clear from the previous derivation. Exactly here comes into play the idea that accelerations have no role in modifying the speed of light.

Applying the acceleration hypothesis Eq. (11) is taken to hold also for a clock on the rim of a disk rotating with velocity v. The general philosophy behind this assumption is that every small portion of the circumference of the rotating platform is instantaneously at rest in a co-moving inertial frame of reference locally "tangent" to the disk and must share its properties. There is excellent experimental evidence that this assumption is correct, e.g. with the CERN muons [13].

A similar reasoning applies to the Lorentz contraction. Consider a rod at rest on the x-axis of the inertial frame S. Let $\Delta x = x_2 - x_1$ and $\Delta x_0 = x_{02} - x_{01}$ describe the rod length with respect to the S and S_0 frames, respectively. If Eq. (4) are written twice, with indices 1 and 2, we can subtract the two sets of equations from one another and get the transformation for space intervals.

$$\Delta x = \Delta x_0 / R \tag{12}$$

Applying once more the acceleration hypothesis, the validity of Eq. (12) can be shown to hold also for a curvilinear rod on the rim of a disk rotating with peripheral velocity v. Moreover, if the curvilinear rod has length exactly the circumference along the platform rim, where we imagine the propagation of light to take place, we can set $\Delta x = L$ and $\Delta x_0 = L_0$, so that Eq. (12) becomes

$$L = L_0 / R \tag{13}$$

This is the Lorentz contraction for the circular rim of the platform. In words: the circumference length of the rotating platform L_0, measured by co-rotating observers, equals the (contracted) circumference length L measured by observers at rest in the laboratory divided by the Lorentz contraction factor R.

6. Sagnac Effect Seen from the Laboratory

We are interested in the application of (11) and (13) to the Sagnac situation. Consider a light source Σ, placed on the disk, emitting two pulses of light in opposite directions. The description of light propagation given by the laboratory observers is the following: two light flashes leave Σ at time $t_0 = 0$. The first one propagates on a circumference, in the sense discordant from the platform rotation, and returns to Σ at time t_{01} after circling around the platform. The second flash propagates on the same circumference, in the sense concordant with the platform rotation, and comes back to Σ at time t_{02} after circling around the platform.

The circular path can be obtained by forcing light to propagate tangentially to the internal surface of a cylindrical mirror. Most textbooks deduce the Sagnac formula (our Eq. (16) below) in the laboratory, but say nothing about the description of the phenomenon given by an observer on the rotating platform: we will see that the theory of special relativity predicts a null effect on the platform, while the inertial transformations give the right answer. For simplicity we will assume that the laboratory is at rest in the privileged frame.

Pulse propagating in the direction opposite to rotation: the disk circumference length L_0 closes with velocity $c + v$. Then

$$t_{01} = \frac{L_0}{c + v} \tag{14}$$

Pulse propagating in the rotational direction: the disk circumference length L_0 closes with velocity $c - v$. Then

$$t_{02} = \frac{L_0}{c - v} \tag{15}$$

From the two previous results it follows

$$\Delta t_0 = t_{02} - t_{01} = \frac{2L_0}{c^2} \frac{v}{1 - v^2/c^2} = \frac{2L_0}{c^2} \frac{v}{R^2} \tag{16}$$

This is essentially the Sagnac formula, very easily deduced in the laboratory (taken to coincide with the privileged system). It is in good agreement with the experimental results.

7. The Sagnac Effect Seen from the Disk

On the disk, we consider only cases of light moving parallel ($\theta = 0$) and antiparallel ($\theta = \pi$) to the local absolute velocity. Then, the inverse velocities of light concordant and discordant with disk rotation have to satisfy Eq. (5) and are respectively given by

$$\frac{1}{c_1(0)} = \frac{1 + \Gamma}{c}; \quad \frac{1}{c_1(\pi)} = \frac{1 - \Gamma}{c} \tag{17}$$

These formulae represent a second application of the acceleration hypothesis: in all ET theories the inverse velocity of light is given by (17) independently of the acceleration of the platform frame. If the circumference length measured on the disk is L we have

$$t_1 = \frac{L}{c_1(0)}; \quad t_2 = \frac{L}{c_1(\pi)} \tag{18}$$

Therefore

$$\Delta t = t_2 - t_1 = \frac{2L\Gamma}{c} \tag{19}$$

This result, unlike (11) and (16), depends on Γ and then on e_1. This is the reason why requiring the consistency of the equations (11) – (16) – (19) gives the right value of e_1.

8. Comparison

The comparison of Δt_0 and Δt should of course take into account the laboratory/disk connection of Eq. (4). By taking the ratio (19)/(16) we get

$$R = \frac{2L\Gamma}{c} \frac{c^2}{2L_0} \frac{R^2}{v} \tag{20}$$

or, if (20) is applied

$$1 = \frac{c}{v}\Gamma \quad \Rightarrow \quad e_1 = 0 \tag{21}$$

This is the anticipated result: only absolute simultaneity allows us to understand the Sagnac effect.

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