Principles of a Frame Indifferent Classical Electromagnetic Field Theory

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In this three part investigation we provide the mathematical foundations and principles of a frame indifferent classical electromagnetic field theory (FIEFT) for arbitrarily moving material media with arbitrary constitution based on convective and comoving time derivative operators. Part 1 is devoted to the mathematical tools utilized in establishing the field theory. It starts with the description of material points in *arbitrary* Euclidean motion, which is a characteristic of rigid (non-deforming) bodies and incompressible inhomogeneous fluids in continuum mechanics. Next we establish the mathematical link between spatial and time derivatives of vector fields between Eulerian and Lagrangian frames via coordinate transformations in Euclidean space. Regarding the images of time derivatives of field quantities, we necessarily invoke the convective and comoving time derivatives. We also provide a proof of the representation of the comoving time derivative for scalar and vector density fields along with its certain differential, commutative and integral properties. In Part 2 we provide the axiomatic structure of our field theory where the frame indifferent electromagnetic field equations are obtained directly as images of Maxwell equations of stationary media under Euclidean (aka observer) transformations. The commutative properties derived between spatial differential and comoving time derivative operators help us derive progressive wave equations for the two standard (translational and rotational) types of Euclidean motion. In Part 3 we describe the general formulation of a boundary value problem for an arbitrarily moving object and investigate three canonical problems of practical interest to demonstrate the predictions of FIEFT.

1. Introduction

The present three part investigation is a preliminary attempt to provide the mathematical foundations and principles of a frame indifferent electromagnetic field theory (FIEFT) of bodies in arbitrary motion. Since the theory is based on the tools and concepts from continuum mechanics, we adopt a tutorial style aiming readers from electrical engineering community with no background in that discipline.

Part 1 is devoted to the kinematical aspects, descriptions and various properties of progressive (convective and comoving time derivative) operators acting on field quantities described in arbitrarily moving material media. We start with the description of material points in arbitrary motion followed by establishing the mathematical link between spatial and time derivatives of vector fields between Eulerian and Lagrangian frames via Euclidean (aka observer) transformations. Regarding the images of time derivatives of field quantities we necessarily invoke the convective and comoving time derivatives. The terms Eulerian and Lagrangian frames that we adopt in this work are well established and called ‘current (or spatial) configuration’ and ‘referential description’, respectively, in fluid mechanics. We also provide a detailed proof of the comoving time derivative for scalar and vector quantities whereas their various commutative properties are also introduced for the first time.

In Part 2 the frame indifferent electromagnetic field equations (FIEFE) are obtained directly as images of Maxwell equations of stationary media with arbitrary constitution under Euclidean transformations. The commutative properties derived for progressive operators help us derive progressive wave equations for media in arbitrary Euclidean motion, which is a characteristic of rigid (non-deforming) bodies and incompressible inhomogeneous fluids. In Part 3 we describe the general formulation of a boundary value problem for an arbitrarily moving object and investigate three canonical problems of practical interest to demonstrate the predictions of FIEFT.

The concept of “material frame indifference” (MFI) and the debates around its alternative interpretations and correct mathematical formulations throughout the history of rational continuum mechanics have been reviewed comprehensively in a recent treatise by M. Frewer [1]. The common motive behind alternatives principle of MIF is that “the structural form and physical content of any physical law (of continuum mechanics) when subject to arbitrary coordinate transformations does not depend on any mathematical quantities which define the geometrical structure of the underlying space-time manifold”. The reflection of this principle in Euclidean space rests on concepts such as “general invariance”, “frame indifference”, “Newtonian space-time”, “Euclidean transformations”, while they clearly contradict with the alternative worldview introduced by Special Relativity Theory of Einstein, which respectively favors “general covariance”, “form invariance”, “Minkowski space-time”, “Lorentz transformations”. The projection of the author’s understanding of MFI onto the structural form of Maxwell equations of classical electromagnetism as described in Part 2 is that the frame indifferent forms of electromagnetic field equations given in (7.2) are obtained from the Maxwell equations of stationary media in (6.1) by a direct substitution of the comoving time derivative operators (introduced in Theorems 4 and 5) in Euclidean frame for the partial time derivatives in Lagrangian frame.

The inspiration behind our present work is the derivation and description of the same set of field equations in (7.2) in vacuum conditions by C.I. Christov in his papers [2-3] (see also the recent review [4]), where the author postulates a direct correspondence between the field quantities of electromagnetism and continuum mechanics in the context of a unifying material worldview of incompressible viscoelastic '*meta-continuum*'.

Throughout the text R_n represents n-dimensional Euclidean space and we employ the abbreviations 'E-' and 'L-' for the frequently used phrases 'Eulerian' and 'Lagrangian', respectively.

PART I: THE MATHEMATICAL TOOLS

2. The Basics of Motion of a Material System

Let us consider a material system filling a domain $D \subset R_3$ whose material points (or matter particles) are in arbitrary motion as observed in a reference laboratory (aka E-) frame $Ox_1x_2x_3t$ as depicted in Fig. 1.

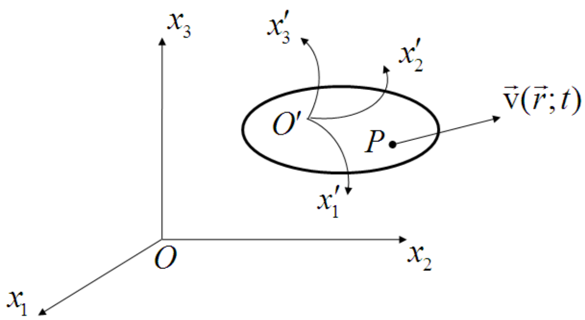

Fig. 1. A material medium in arbitrary motion with linear velocity $\vec{v}(\vec{r};t)$.

In E-frame we shall denote the instantaneous coordinates (or trajectory) of a material point (or a matter particle) P by the position vector $\vec{r}_P(t) = [x_1(t), x_2(t), x_3(t)]$. In terms of the arbitrary instantaneous velocity vector field $\vec{v}(\vec{r};t)$, the coordinates of the matter particles at time t can be specified through the initial (Cauchy) boundary value problem

$$\begin{cases} d\vec{r}_P / dt = \vec{v}(\vec{r}_P;t) \\ \vec{r}_P(t=t_0) = \vec{r}_0 \ \text{(fixed)} \end{cases} \tag{2.1}$$

The instantaneous property of the velocity field provides it independent of the reference time t_0 (*cf* [5, Property 1.1]). Next we introduce the local arbitrary curvilinear reference (aka L-) frame $O'x_1'x_2'x_3't'$ for any material point P in which its location, say $\vec{r}_P'$, is assumed unaltered while the frame is in motion with respect to E-frame. One may refer to [6] regarding the discussions on the construction of Cauchy problems for the instantaneous location of material points.

To emphasize on the distinction between mechanical and relativistic theories we shall introduce the following postulate regarding the two inertial reference frames.

Postulate 1: *Regardless of the arbitrary instantaneous linear velocity vector of L-frame with reference to E-frame no time dilation or length contraction is assumed in the context of FIEFT between any measurement taken by ideal devices considered fixed in these two frames.*

The postulate of temporal and spatial invariance treats time as a nonphysical quantity

$$t = t' \tag{2.2a}$$

and the Euclidean metric measured in the two reference frames as the same. Mathematically, if P_1 and P_2 are two material points with instantaneous Eulerian & Lagrangian position vectors $\vec{r}_{P_1}$ & $\vec{r}_{P_1}'$ and $\vec{r}_{P_2}$ & $\vec{r}_{P_2}'$ in E- & L- frames, then one assumes

$$\left|\vec{r}_{P_1}(t_1) - \vec{r}_{P_2}(t_2)\right| = \left|\vec{r}_{P_1}'(t_1') - \vec{r}_{P_2}'(t_2')\right| = \left|\vec{r}_{P_1}'(t_1) - \vec{r}_{P_2}'(t_2)\right| \tag{2.2b}$$

3. The Convective Derivative

Theorem 1: (Convective Derivative in R_3)
Consider an arbitrary field quantity $g(\vec{r};t)$ (scalar, vector or tensor), in an arbitrary medium traveling with a linear instantaneous velocity field $\vec{v}(\vec{r};t)$ w.r.t. the E-frame in R_3. The time rate of change experienced in the L-frame is called the 'convective' (aka L-, substantial, total time, Euler's material) derivative, and has the form

$$\frac{D}{Dt} g(\vec{r};t) = \frac{\partial}{\partial t} g(\vec{r};t) + \vec{v}(\vec{r};t) \cdot \operatorname{grad} g(\vec{r};t) \tag{3.1}$$

Proof: The convective derivative of an arbitrary field quantity $g(\vec{r};t)$ in L-frame in R_3 has the form

$$\frac{D}{Dt} g(\vec{r};t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[g(\vec{r}(t+\Delta t);t+\Delta t) - g(\vec{r}(t);t) \right]$$

where $\vec{r}(t+\Delta t) = \vec{r}(t) + (\Delta t)\vec{v}(\vec{r};t)$. The Taylor series of expansion of $g(\vec{r}(t+\Delta t);t+\Delta t)$ around $(\vec{r}(t);t)$ can be written as

$$\begin{aligned} g(\vec{r}(t+\Delta t);t+\Delta t) &= g(\vec{r}(t);t) + (\Delta t)\left(\sum_{i=1}^{3} \frac{\partial g}{\partial x_i}\frac{\partial x_i}{\partial t} + \frac{\partial g}{\partial t} \right) + \mathrm{o}(\Delta t) \\ &= g(\vec{r};t) + \Delta t \left[\vec{v} \cdot \operatorname{grad} g(\vec{r};t) + \frac{\partial}{\partial t} g(\vec{r};t) \right] + \mathrm{o}(\Delta t) \end{aligned}$$

and placed into the limit definition to yield the desired result.

3.1. Certain Differential Properties of the Convective Derivative in R_3

Let c, $\vec{C}$ be constant scalar/vector quantities and the scalar/vector fields $f(\vec{r};t)$, $g(\vec{r};t)$, $\vec{A}(\vec{r};t)$, $\vec{B}(\vec{r};t)$ be of $C^1(R_3)$. Based on the linearity of the convective derivative operator one can observe the following properties.

Property C1: $\frac{D}{Dt}c = 0$, $\frac{D}{Dt}\vec{C} = \vec{0}$

Property C2: $\frac{D}{Dt}(cf) = c\frac{Df}{Dt}$, $\frac{D}{Dt}(c\vec{A}) = c\frac{D\vec{A}}{Dt}$

Property C3: $\frac{D}{Dt}(f \pm g) = \frac{Df}{Dt} \pm \frac{Dg}{Dt}$, $\frac{D}{Dt}(\vec{A} \pm \vec{B}) = \frac{D\vec{A}}{Dt} \pm \frac{D\vec{B}}{Dt}$

Property C4: $\frac{D}{Dt}(fg) = \frac{Df}{Dt}g + f\frac{Dg}{Dt}$

Property C5: $\frac{D}{Dt}(f\vec{A}) = \frac{Df}{Dt}\vec{A} + f\frac{D\vec{A}}{Dt}$

Property C6: $\frac{D}{Dt}(\vec{A} \cdot \vec{B}) = \frac{D\vec{A}}{Dt} \cdot \vec{B} + \vec{A} \cdot \frac{D\vec{B}}{Dt} + \vec{A} \times \operatorname{curl}\vec{B} + \vec{B} \times \operatorname{curl}\vec{A}$

Property C7: $\frac{D}{Dt}(\vec{A} \times \vec{B}) = \frac{D\vec{A}}{Dt} \times \vec{B} + \vec{A} \times \frac{D\vec{B}}{Dt}$

where we incorporate the vector identity

$$\mathrm{grad}\left(\vec{A}\times\vec{B}\right)=\left(\mathrm{grad}\vec{A}\right)\times\vec{B}-\left(\mathrm{grad}\vec{B}\right)\times\vec{A}$$

Property C8: $\frac{D}{Dt}\left(f^{n}\right)=nf^{n-1}\frac{Df}{Dt}\;,\;n\in\mathbb{R}$

3.2. Convective Derivative on a Surface

Let (u_1,u_2) be the real valued parametric curves of a two-sided, regular surface $S(t)$ described by the position vector $\vec{r}_S=\vec{r}(u_1,u_2)$. A quantity that assumes one or more definite values at each point of a surface is called a 'density function' for the surface. Let us consider a scalar density function $\psi(u_1,u_2)$ and a vector density function

$$\vec{A}(u_1,u_2)=A_1(u_1,u_2)\hat{u}_1+A_2(u_1,u_2)\hat{u}_2+A_n(u_1,u_2)\hat{n}\quad,$$

where $\hat{u}_1,\hat{u}_2$ are unit tangent vectors along the curves $u_1=\mathrm{const.}$ and $u_2=\mathrm{const.}$ and $\hat{n}(t)$ is the unit normal of $S(t)$, which constitute a right handed system. Then it can be shown that the gradient, divergence and curl operators acting on the density functions on a surface are as follows:

Theorem 2: (Surficial Vector Differential Operators)

$$\mathrm{grad}_S\psi=\frac{\hat{u}_1}{h_1}\frac{\partial\psi}{\partial u_1}+\frac{\hat{u}_2}{h_2}\frac{\partial\psi}{\partial u_2}\tag{3.2a}$$

$$\mathrm{div}_S\vec{A}=\frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_1)+\frac{\partial}{\partial u_2}(h_1A_2)\right]-2\Omega A_n\tag{3.2b}$$

$$\begin{aligned}\mathrm{curl}_S\vec{A}=&\frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_2)-\frac{\partial}{\partial u_2}(h_1A_1)\right]\hat{n}\\&+\frac{A_2}{\alpha_2}\hat{u}_1-\frac{A_1}{\alpha_1}\hat{u}_2+\mathrm{grad}_SA_n\times\hat{n}\end{aligned}\tag{3.2c}$$

Here h_1, h_2 are the metric coefficients of the parametric curves; the principle radii of curvature $\alpha_{1,2}$ are related to the metric coefficients through

$$\frac{1}{\alpha_1}=-\frac{1}{h_1}\frac{dh_1}{dn}\;,\;\frac{1}{\alpha_2}=-\frac{1}{h_2}\frac{dh_2}{dn}\;;\tag{3.2d,e}$$

and

$$2\Omega=\frac{1}{\alpha_1}+\frac{1}{\alpha_2}=-\mathrm{div}_S(\hat{n})\tag{3.2f}$$

is called the first curvature of $S(t)$. A proof of Theorem 2 can be seen in [7], [8 Ch. 12] as one of the earliest accounts, where the density function is termed as 'point function'.

Lemma 1: $\mathrm{grad}_S(\hat{n})=-\frac{1}{\alpha_1}\hat{u}_1\hat{u}_1-\frac{1}{\alpha_2}\hat{u}_2\hat{u}_2$

Proof: One substitutes the Gauss-Codazzi formulas (*cf.* [8, Ch.5])

$$\begin{aligned}\frac{\partial\hat{n}}{\partial u_1}&=\frac{\partial}{\partial u_1}\left(\hat{u}_1\times\hat{u}_2\right)\\&=\frac{\partial\hat{u}_1}{\partial u_1}\times\hat{u}_2+\hat{u}_1\times\frac{\partial\hat{u}_2}{\partial u_1}=\frac{h_1}{\alpha_1}\hat{n}\times\hat{u}_2+\vec{0}=-\frac{h_1}{\alpha_1}\hat{u}_1\end{aligned}$$

$$\begin{aligned}\frac{\partial\hat{n}}{\partial u_2}&=\frac{\partial}{\partial u_2}\left(\hat{u}_1\times\hat{u}_2\right)\\&=\frac{\partial\hat{u}_1}{\partial u_2}\times\hat{u}_2+\hat{u}_1\times\frac{\partial\hat{u}_2}{\partial u_2}=\vec{0}+\frac{h_2}{\alpha_2}\hat{u}_1\times\hat{n}=-\frac{h_2}{\alpha_2}\hat{u}_2\end{aligned}$$

in the definition $\mathrm{grad}_S(\hat{n})=\frac{\hat{u}_1}{h_1}\frac{\partial\hat{n}}{\partial u_1}+\frac{\hat{u}_2}{h_2}\frac{\partial\hat{n}}{\partial u_2}$ to obtain the desired result directly.

Corollary 1: (Convective Derivative on a Surface)

The restriction of the definition of the convective derivative in R_3 to R_2 directly yields its expression on a surface. In this case, in virtue of (3.2) the time rate of change of a scalar field $f(\vec{r}_S;t)$ or a vector field $\vec{A}(\vec{r}_S;t)$ defined only on a two-sided, regular surface $S(t)$ traveling with a linear instantaneous velocity $\vec{\mathrm{v}}(\vec{r}_S;t)$ as experienced in L-frame reads

$$\frac{D}{Dt}f(\vec{r}_S;t)=\frac{\partial}{\partial t}f(\vec{r}_S;t)+\vec{\mathrm{v}}\cdot\mathrm{grad}_Sf(\vec{r}_S;t)\tag{3.3a}$$

$$\frac{D}{Dt}\vec{A}(\vec{r}_S;t)=\frac{\partial}{\partial t}\vec{A}(\vec{r}_S;t)+\vec{\mathrm{v}}\cdot\mathrm{grad}_S\vec{A}(\vec{r}_S;t)\tag{3.3b}$$

As a result all the properties C1 - C8 also apply for the surficial convective derivative in (3.3).

According to an L-observer the surface is, by definition, at rest with a fixed unit normal. For the E-observer this brings along the following property:

Property C9: $\frac{D\hat{n}}{Dt}=\vec{0}$, $\forall t$, where $\hat{n}(t)$ is the unit normal of a regular surface $S(t)$ in arbitrary motion.`

Then in virtue of Property C5, for an arbitrary scalar point function ψ one reaches at the following property.

Property C10: $\frac{D}{Dt}\left(\psi\hat{n}\right)=\hat{n}\frac{D\psi}{Dt}$, where $\hat{n}(t)$ is the unit normal of a regular surface $S(t)$ in arbitrary motion and $\psi(\vec{r}_S;t)$ is an arbitrary density function.

The results obtained in this section can also be extended for "curvilineal" and convective differential operators for density functions defined on space curves, which shall be omitted.

4. Images under Euclidean Transformations

Assumption 1: Let the general coordinate transformations between E- and L- frames be given by the sets

$$x_i'=f_i'(x_j;t)\;\;,\;\;x_i=f_i(x_j';t)\;\;,\;\;i,j=1,2,3\tag{4.1a,b}$$

where $\{x_i\}$ and $\{x_i'\}$ correspond to the Cartesian E-coordinates and to the general curvilinear L-coordinates, respectively. We assume the maps $f_i,f_i'\in C^2(R_3)$ bijective, not necessarily linear and provide an admissible change of coordinates locally in the moving material medium D. Our current investigation is restricted to Euclidean (aka observer) transformations in the form

$$\vec{r}'=\vec{c}(t)+\bar{\bar{Q}}(t)\cdot\vec{r}\;\;,\;\;\vec{r}=\bar{\bar{Q}}^{TR}(t)\cdot\left[\vec{r}'-\vec{c}(t)\right]\tag{4.2a,b}$$

where $\bar{\bar{Q}}(t)$ is an arbitrary time dependent orthogonal tensor with the superscript TR representing its transpose. Euclidean transformation is a generalization of Galilean transformation involving time dependence in the translation vector $\vec{c}(t)$ and the rotation matrix $\bar{\bar{Q}}(t)$. The orthonormal bases $\hat{r}=(\hat{x}_1,\hat{x}_2,\hat{x}_3)$ and $\hat{r}'=(\hat{x}_1',\hat{x}_2',\hat{x}_3')$ are also transformed as

$$\hat{r}' = \bar{\bar{Q}}(t)\cdot\hat{r} \quad , \quad \hat{r} = \bar{\bar{Q}}^{TR}(t)\cdot\hat{r}' \qquad (4.2c,d)$$

Assumption 2: Let the linear velocity vector field $\vec{v}(x_i;t)$ of the moving material medium be expressible as a contravariant vector with contravariant components v^i in $\{x_i\}$ frame.

This assumption conforms to the nature of the motion of a point particle along an arbitrary path. One verification can be found at [9, Ex. 3.4]).

4.1. Special Case 1: Translational Motion

In this simplest for which each material point in D has the same velocity vector $\vec{v}(\vec{r};t) = \vec{v}(t)$ one may specify the L-frame as Cartesian and parallel to E-frame with $\hat{x}'_i = \hat{x}_i$, $i = 1,2,3$, which reads

$$x'_i = x_i - \int_{-\infty}^{t} v^i(\xi)\,d\xi \quad , \quad x_i = x'_i + \int_{-\infty}^{t} v^i(\xi)\,d\xi \qquad (4.3a,b)$$

$$v^i(t) = \hat{x}'_i \cdot \vec{v}(t) = \hat{x}_i \cdot \vec{v}(t) \quad , \quad i = 1,2,3 . \qquad (4.3c)$$

4.2. Special Case 2: Rotational Motion

We may consider the medium D in rotational motion around x_3 axis with an arbitrary angular velocity $\omega(t)$ as depicted in Fig. 2. In this case the instantaneous linear velocity of any material point P can be expressed in cylindrical coordinates (ρ,ϕ,x_3) with unit vectors $(\hat{\rho},\hat{\phi},\hat{x}_3)$ by

$$\vec{v}(\vec{r};t) = \omega(t)\rho\hat{\phi}(t) = \omega(t)\rho\left[-\hat{x}_1\sin\phi(t) + \hat{x}_2\cos\phi(t)\right] \qquad (4.4a)$$

along with the set of transformations $x'_3 = x_3$ and

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} \cos\phi(t) & \sin\phi(t) \\ -\sin\phi(t) & \cos\phi(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (4.4b)$$

where

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} \omega(\zeta)\,d\zeta \quad . \qquad (4.4c)$$

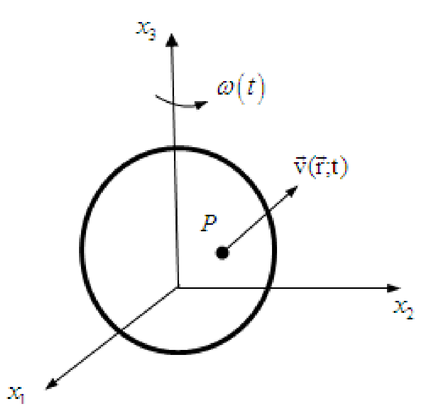

Fig. 2. Rotational motion of an arbitrary material medium around a fixed axis with velocity $\vec{v}(\vec{r};t)$.

Definition 1: (Images of Domains)

Let the domain occupied by a material system be denoted in the general forms $D' = \{(\vec{r}';t) \,|\, a(t) \le \alpha'(\vec{r}';t) \le b(t)\} \subset R_3$ and $D = \{(\vec{r};t) \,|\, a(t) \le \alpha(\vec{r};t) \le b(t)\} \subset R_3$ in L- and E-frames, respectively, under the general transformations

$$\alpha(\vec{r};t) = \alpha'(\vec{r}';t)\Big|_{x'_i = f'_i(x_j;t)} \quad , \quad g(\vec{r};t) = g'(\vec{r}';t)\Big|_{x'_i = f'_i(x_j;t)} .$$

Then one calls D (or D') as the image of D' (or D) under the bijective coordinate transformation f'_i (or f_i) and may express it symbolically as

$$D' \xrightarrow{f'_i} D \quad , \quad D \xrightarrow{f_i} D' \text{ or simply by } D' \rightleftarrows D .$$

Definition 2: (*Objective Fields*)

Let arbitrary (smooth enough) scalar, vector and tensor valued density fields in medium D be denoted by $g(\vec{r};t)$, $\vec{A}(\vec{r};t)$ and $\bar{\bar{T}}(\vec{r};t)$, respectively. Regarding $\bar{\bar{T}}(\vec{r};t)$, we assume it a general contravariant tensor of every order. If these field quantities preserve their physical state in both frames of reference, then the coordinate transformations in Assumption 1 are called passive transformations and the density fields are called objective (or frame indifferent) fields. One may refer to [1, Sect. 6] for further information about the classifications of variable transformations. In this case the images of the density fields in the L-frame are described with the maps

$$\left\{g'(\vec{r}';t), \vec{A}'(\vec{r}';t), \bar{\bar{T}}'(\vec{r}';t)\right\} = \left\{g(\vec{r};t), \vec{A}(\vec{r};t), \bar{\bar{T}}(\vec{r};t)\right\}\Big|_{x_i = f_i(x'_j;t)}$$

or symbolically as

$$\left\{g', \vec{A}', \bar{\bar{T}}'\right\} \xrightarrow{f_i} \left\{g, \vec{A}, \bar{\bar{T}}\right\}$$

and vice versa. In case of Euclidean transformations as in (4.2) the exact mathematical relations that ensure the invariance of the directions of vector and tensor fields are given by (cf. [10, Sect. 4.3], [11, Sec.II.2]).

$$g'(\vec{r}';t) = g(\vec{r};t) \quad , \quad \vec{A}'(\vec{r}';t) = \bar{\bar{Q}}(t)\cdot\vec{A}(\vec{r};t) \qquad (4.5a)$$

$$\bar{\bar{T}}'(\vec{r}';t) = \bar{\bar{Q}}(t)\cdot\bar{\bar{T}}(\vec{r};t)\cdot\bar{\bar{Q}}^{TR}(t)$$

Theorem 3: (Image of Spatial Derivatives)

The action of vector differential operators on (smooth enough) objective scalar, vector and tensor valued density fields $g(\vec{r};t)$, $\vec{A}(\vec{r};t)$, $\bar{\bar{T}}(\vec{r};t)$ yield objective density fields described by the Euclidean transformations

$$\begin{aligned}
\mathrm{grad}'\, g'(\vec{r}';t) &= \bar{\bar{Q}}(t)\cdot \mathrm{grad}\, g(\vec{r};t) \\
\mathrm{div}'\vec{A}'(\vec{r}';t) &= \mathrm{div}\vec{A}(\vec{r};t) \\
\mathrm{div}'\bar{\bar{T}}'(\vec{r}';t) &= \bar{\bar{Q}}(t)\cdot \mathrm{div}\bar{\bar{T}}(\vec{r};t) \\
\mathrm{curl}'\vec{A}'(\vec{r}';t) &= \bar{\bar{Q}}(t)\cdot \mathrm{curl}\vec{A}(\vec{r};t) \\
\mathrm{curl}'\bar{\bar{T}}'(\vec{r}';t) &= \bar{\bar{Q}}(t)\cdot \mathrm{curl}\bar{\bar{T}}(\vec{r};t) \\
\mathrm{lap}' g'(\vec{r}';t) &= \mathrm{lap} g(\vec{r};t) \\
\mathrm{lap}'\vec{A}'(\vec{r}';t) &= \bar{\bar{Q}}(t)\cdot \mathrm{lap}\vec{A}(\vec{r};t)
\end{aligned} \qquad (4.5b)$$

or symbolically by

$$\mathrm{grad}'\, g' \rightleftarrows \mathrm{grad}\, g$$

$$\mathrm{div}'\vec{A}' \rightleftarrows \mathrm{div}\vec{A} \;,\; \mathrm{div}'\overline{\overline{T}}' \rightleftarrows \mathrm{div}\overline{\overline{T}}$$

$$\mathrm{curl}'\vec{A}' \rightleftarrows \mathrm{curl}\vec{A} \;,\; \mathrm{curl}'\overline{\overline{T}}' \rightleftarrows \mathrm{curl}\overline{\overline{T}}$$

$$\mathrm{lap}'g' \rightleftarrows \mathrm{lap}g \;,\; \mathrm{lap}'\vec{A}' \rightleftarrows \mathrm{lap}\vec{A} \;.$$

Here, "lap" represents the scalar/vector Laplacian operator. A proof for gradient and divergence operators is available at [10, Sect. 4.3], while the rest of the results can also be obtained in a straightforward manner.

Definition 3: (The Comoving Time Derivative of a Tensor Density Field in R_3)

The comoving time derivative of an arbitrary (smooth enough) contravariant tensor density field $g(\vec{r};t)$ as observed in E-frame is described by

$$\frac{\Diamond}{\Diamond t} g(\vec{r};t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[g'(\vec{r}';t+\Delta t) - g'(\vec{r}';t)\right] \tag{4.6}$$

with the assumption $g(\vec{r};t) = g'(\vec{r}';t)$ at time t.

Corollary 2: The comoving time derivative (4.6) can be addressed as the image of partial time derivative operator in L- frame; i.e.,

$$\frac{\Diamond}{\Diamond t} g(\vec{r};t) \rightleftarrows \frac{\partial g'}{\partial t}(\vec{r}';t) \tag{4.7}$$

It is also known as 'the upper convected material derivative' or 'Oldroyd derivative' in continuum mechanics when tensor density fields are concerned and is the only member of a family of invariant time derivatives (*cf.* [12]) that correctly postulates field equations not only in continuum mechanics but also the electromagnetism of moving bodies (*cf.* [2-4]). The Oldroyd derivative was introduced in [13] for establishing invariant forms of rheological equations of state for a homogeneous continuum, suitable for application to all conditions of motion and stress, particularly when the frame of reference is a coordinate system convected with the material. In that sense the comoving time derivative of scalar/vector density fields can be interpreted as the Oldroyd derivative of a tensor of rank zero/one. In the context of electrical engineering we shall prefer the terminology 'comoving time derivative' to 'Oldroyd derivative' since the latter is rather established in Continuum Mechanics and essentially related with tensor quantities.

The comoving time derivative accounts for taking the directional derivative of covariant tensor density field $g(\vec{r};t)$ along the covariant velocity vector $\vec{\mathrm{v}}(\vec{r};t)$ of the moving material medium, which is a generalization of the adjective (i.e., nonlinear) part of the usual convective derivative in (3.1) when $\vec{\mathrm{v}}(\vec{r};t) \neq \vec{\mathrm{v}}(t)$. It is a generalization of the Lie derivative along a curve (aka the world-line of a point) in 4-D manifold called Galilean space-time. It is a natural derivative operator following a material point in arbitrary motion and related to the Lie derivative by the general relation

$$\frac{\Diamond}{\Diamond t} g(\vec{r};t) = \frac{\partial}{\partial t} g(\vec{r};t) + L_{\vec{\mathrm{v}}} g(\vec{r};t) \tag{4.8}$$

The partial time derivative at right hand side in (4.8) accounts for the changes of the components as functions of time and its complement Lie derivative represent the changes due to the fact that the coordinate system and the associated bases are also changing with time (being 'convected' with the velocity field of the moving material medium).

For more information and geometrical interpretations of Lie and comoving time derivatives one may refer to [14].

Theorem 4: (Comoving Time Derivative of Scalar Density Field)

The comoving time derivative of a scalar density field $g(\vec{r};t) \in C^1(R_3)$ in a material medium moving with an arbitrary linear instantaneous velocity field $\vec{\mathrm{v}}(\vec{r};t)$ is calculated as

$$\begin{aligned}\frac{\Diamond}{\Diamond t} g &= \frac{D}{Dt} g + g(\mathrm{div}\vec{\mathrm{v}}) \\ &= \frac{\partial}{\partial t} g + \vec{\mathrm{v}} \cdot \mathrm{grad} g + g(\mathrm{div}\vec{\mathrm{v}}) \\ &= \frac{\partial}{\partial t} g + \mathrm{div}(\vec{\mathrm{v}} g)\end{aligned} \tag{4.9a}$$

where

$$L_{\vec{\mathrm{v}}} g = \mathrm{div}(\vec{\mathrm{v}} g) = \vec{\mathrm{v}} \cdot \mathrm{grad}\, g + f(\mathrm{div}\, g) \tag{4.9b}$$

stands for the Lie derivative of a scalar density field $g(\vec{r};t)$.

Theorem 5: (Comoving Time Derivative of Vector Density Field)

The comoving time derivative of a vector field $\vec{A}(\vec{r};t) \in C^1(R_3)$ in a material medium moving with an arbitrary linear instantaneous velocity field $\vec{\mathrm{v}}(\vec{r};t)$ is calculated as

$$\frac{\Diamond}{\Diamond t}\vec{A} = \frac{\partial}{\partial t}\vec{A} + \vec{\mathrm{v}} \cdot \mathrm{grad}\vec{A} - \vec{A} \cdot (\mathrm{grad}\vec{\mathrm{v}}) + \vec{A}(\mathrm{div}\vec{\mathrm{v}}) \tag{4.10a}$$

where

$$L_{\vec{\mathrm{v}}}\vec{A} = \vec{\mathrm{v}} \cdot \mathrm{grad}\,\vec{A} - \vec{A} \cdot (\mathrm{grad}\,\vec{\mathrm{v}}) + \vec{A}(\mathrm{div}\,\vec{\mathrm{v}}) \tag{4.10b}$$

stands for the Lie derivative of a vector density field $\vec{A}(\vec{r};t)$.

By help of vector identities the comoving time derivatives can also be written in the following alternative forms:

$$\begin{aligned}\frac{\Diamond}{\Diamond t}\vec{A} &= \frac{\partial}{\partial t}\vec{A} + \mathrm{div}(\vec{\mathrm{v}}\vec{A}) - \vec{A} \cdot (\mathrm{grad}\,\vec{\mathrm{v}}) \\ &= \frac{D}{Dt}\vec{A} - \vec{A} \cdot (\mathrm{grad}\,\vec{\mathrm{v}}) + \vec{A}(\mathrm{div}\vec{\mathrm{v}}) \\ &= \frac{\partial}{\partial t}\vec{A} + \vec{\mathrm{v}}\,\mathrm{div}\,\vec{A} - \mathrm{curl}(\vec{\mathrm{v}} \times \vec{A})\end{aligned} \tag{4.11}$$

4.3. Proofs of Theorems 4 and 5:

A derivation of (4.10a) based on its limit definition can be found in [2,3] and in sufficient detail in [15, Sec.4.4]. In this section we will provide the proofs of both (4.9a) and (4.10a) in a unified manner following a slightly different approach.

Consider a tensor density field whose volume integral in an arbitrarily moving material medium describes a field quantity (such as total charge or mass), the physical nature and quantity of which is assumed the same for all observers in Euclidean space. To be specific let $D(t)$, $D'(t)$ and g, g' be the representations of the same moving material medium and the tensor density field in E- and L-frames, respectively. Furthermore, let the Cartesian E-coordinates $\{x_i\}$ coincide with the general curvilinear L-coordinates $\{x_i'\}$ at time t, which requires the field quantities and the differential volume elements to be the same at that instant:

$$d\vartheta = d\vartheta' \quad , \quad g(\vec{r};t) = g'(\vec{r}';t) \quad \text{(at time } t\text{)} \tag{4.12a,b}$$

Then, after an infinitesimal period Δt, the medium and the tensor densities are denoted by $D(t+\Delta t), D'(t+\Delta t)$ and $g(\vec{r};t+\Delta t)$, $g'(\vec{r}';t+\Delta t)$ as depicted in Fig. 3.

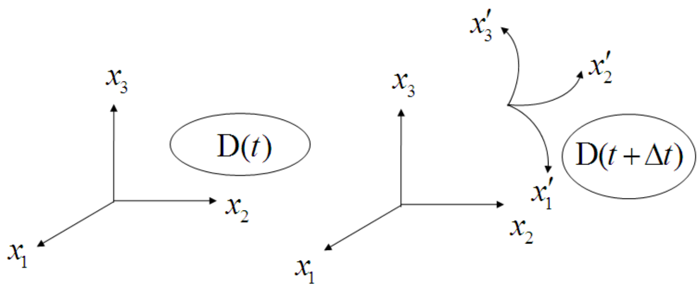

Fig. 3. E- and L- frames of the moving material medium at times t and $t+\Delta t$

At time $t+\Delta t$ the coordinate transformations between the two systems can be given by first order as

$$x_j = x_j' + \mathrm{v}^j(x_i;t)\,\Delta t\,,\; x_i' = x_i - \mathrm{v}^i(x_j;t)\,\Delta t\,,\; i,j=1,2,3 \tag{4.13a,b}$$

Spatial partial differentiations in (4.13) yield

$$\frac{\partial x_j}{\partial x_i'} = \delta_i^j + \Delta t\frac{\partial \mathrm{v}^j}{\partial x_i} + \mathrm{o}(\Delta t)\,,\; \frac{\partial x_i'}{\partial x_j} = \delta_j^i - \Delta t\frac{\partial \mathrm{v}^i}{\partial x_j} + \mathrm{o}(\Delta t)\,, \tag{4.14a,b}$$

where $\delta_i^j = \begin{cases} 1, i=j \\ 0, i\neq j \end{cases}$ denotes the Kronecker delta. Since the medium is dynamic, we cannot talk about the validity of (4.12) also at time $t+\Delta t$. Instead, the only conclusive statement one can do at time $t+\Delta t$ is the invariance (or conservation) of the integral

$$\int_{D(t+\Delta t)} g(\vec{r};t+\Delta t)\,d\vartheta = \int_{D'(t+\Delta t)} g'(\vec{r}';t+\Delta t)\,d\vartheta' \tag{4.15}$$

An application of the property (4.15) can be found at [16, Ch.2] in a different context. At time $t+\Delta t$, the differential volume elements are connected by

$$d\vartheta = J\,d\vartheta' \tag{4.16a}$$

where

$$J = \det\left\|\frac{\partial x_i}{\partial x_j'}\right\| = 1 + \Delta t\frac{\partial \mathrm{v}^i}{\partial x_i} + \mathrm{o}(\Delta t) = 1 + (\Delta t)\mathrm{div}\vec{\mathrm{v}} + \mathrm{o}(\Delta t) \tag{4.16b}$$

denotes the nonzero Jacobian of the transformation matrix (aka the deformation gradient). Substituting (4.16a) into (4.15) yields

$$\int_{D'(t+\Delta t)} g(\vec{r};t+\Delta t)\,J\,d\vartheta' = \int_{D'(t+\Delta t)} g'(\vec{r}';t+\Delta t)\,d\vartheta' \tag{4.17a}$$

which, for an arbitrary material medium, necessitates the transformation rule

$$g'(\vec{r}';t+\Delta t) = J g(\vec{r};t+\Delta t) \tag{4.17b}$$

A proof for formal equivalences as in (4.17b) can be seen in [10, p. 42] (see also [17, p. 3]). We further consider the Taylor series expansion of $g(\vec{r};t+\Delta t)$ around time t as

$$g(\vec{r};t+\Delta t) = g(\vec{r};t) + \Delta t\frac{D}{Dt}g(\vec{r};t) + o(\Delta t) \tag{4.18a}$$

When $g(\vec{r};t)$ is a scalar field, substituting (4.18a) and (4.12b) into (4.17b) one gets

$$\begin{aligned} &g'(\vec{r}';t+\Delta t) - g'(\vec{r}';t) \\ &= (J-1)g'(\vec{r}';t) + J(\Delta t)\frac{D}{Dt}g(\vec{r};t) + \mathrm{o}(\Delta t) \\ &= (\Delta t)\left[(\mathrm{div}\vec{\mathrm{v}})g(\vec{r};t) + \frac{D}{Dt}g(\vec{r};t)\right] + \mathrm{o}(\Delta t) \end{aligned} \tag{4.18b}$$

and its comoving time derivative (4.6) can be obtained in virtue of (4.18b) directly as (4.9a).

When $g(\vec{r};t)$ is a (contravariant) vector field as $\vec{A}(\vec{r};t)$, the relation (4.17b) can be written in terms of its contravariant components as

$$\begin{aligned} &A^{i'}(\vec{r}';t+\Delta t) = J\frac{\partial x_i'}{\partial x_j}A^j(\vec{r};t+\Delta t) \\ &= \left(1 + (\Delta t)\mathrm{div}\vec{\mathrm{v}} + o(\Delta t)\right)\left(\delta_j^i - \Delta t\frac{\partial \mathrm{v}^i}{\partial x_j} + \mathrm{o}(\Delta t)\right)A^j(\vec{r};t+\Delta t) \\ &= A^i(\vec{r};t+\Delta t) + (\Delta t)\left[(div\vec{\mathrm{v}})A^i(\vec{r};t) - \frac{\partial \mathrm{v}^i}{\partial x_j}A^j(\vec{r};t)\right] + \mathrm{o}(\Delta t) \\ &= A^i(\vec{r};t) \\ &+ (\Delta t)\left[\frac{D}{Dt}A^i(\vec{r};t) + (\mathrm{div}\vec{\mathrm{v}})A^i(\vec{r};t) - \left(\mathrm{grad}\mathrm{v}^i\right)\cdot\vec{A}(\vec{r};t)\right] \\ &+ \mathrm{o}(\Delta t) \end{aligned} \tag{4.19a}$$

The relation (4.19a) provides the connection between the contravariant components of $\vec{A}$ at times t and $t+\Delta t$. Multiplying each side by the unit vectors $\hat{x}_i'$ and $\hat{x}_i$ and using (4.12b), it can be arranged as

$$\begin{aligned} &\vec{A}'(\vec{r}';t+\Delta t) - \vec{A}'(\vec{r}';t) = \\ &(\Delta t)\left[\frac{D}{Dt}\vec{A}(\vec{r};t) + (\mathrm{div}\vec{\mathrm{v}})\vec{A}(\vec{r};t) - \vec{A}(\vec{r};t)\cdot\mathrm{grad}\vec{\mathrm{v}}\right] + \mathrm{o}(\Delta t) \end{aligned} \tag{4.19b}$$

Finally, (4.19b) can be placed into (4.6) to get the desired relation (4.10a). The relations (4.16) for the deformation of a volume are well known in continuum mechanics and can be found in many standard textbooks (*cf.* [18]).

Theorem 6: (Image of Time Derivative)

The action of the comoving time derivative operator on objective scalar, vector and tensor valued density fields $g(\vec{r};t)$, $\vec{A}(\vec{r};t)$, $\bar{\bar{T}}(\vec{r};t)$ of $C^1(R_3)$ yield objective density fields described by the Euclidean transformations

$$\frac{\partial}{\partial t}g'(\vec{r}';t) = \frac{\lozenge}{\lozenge t}g(\vec{r};t) \tag{4.20a}$$

$$\frac{\partial}{\partial t}\vec{A}'(\vec{r}';t) = \bar{\bar{Q}}(t)\cdot\frac{\lozenge}{\lozenge t}\vec{A}(\vec{r};t) \tag{4.20b}$$

$$\frac{\partial}{\partial t}\bar{\bar{T}}'(\vec{r}';t) = \bar{\bar{Q}}(t)\cdot\frac{\lozenge}{\lozenge t}\bar{\bar{T}}(\vec{r};t)\cdot\bar{\bar{Q}}^{TR}(t) \tag{4.20c}$$

or symbolically by

$$\frac{\partial}{\partial t} g'(\vec{r}';t) \rightleftarrows \frac{\lozenge}{\lozenge t} g(\vec{r};t)$$

$$\frac{\partial}{\partial t} \vec{A}'(\vec{r}';t) \rightleftarrows \frac{\lozenge}{\lozenge t} \vec{A}(\vec{r};t)$$

$$\frac{\partial}{\partial t} \overline{\overline{T}}'(\vec{r}';t) \rightleftarrows \frac{\lozenge}{\lozenge t} \overline{\overline{T}}(\vec{r};t)$$

A proof is available at [10, Sec.4.3].

Lemma 2: The relations (4.20) can be generalized for an arbitrary order of differentiation $k \geq 1$ as

$$\frac{\partial^k}{\partial t^k} g'(\vec{r}';t) = \frac{\lozenge^k}{\lozenge t^k} g(\vec{r};t) \tag{4.21a}$$

$$\frac{\partial^k}{\partial t^k} \vec{A}'(\vec{r}';t) = \overline{\overline{Q}}(t) \cdot \frac{\lozenge^k}{\lozenge t^k} \vec{A}(\vec{r};t) \tag{4.21b}$$

$$\frac{\partial^k}{\partial t^k} \overline{\overline{T}}'(\vec{r}';t) = \overline{\overline{Q}}(t) \cdot \frac{\lozenge^k}{\lozenge t^k} \overline{\overline{T}}(\vec{r};t) \cdot \overline{\overline{Q}}^{TR}(t) \tag{4.21c}$$

A combination of the results (4.5) and (4.21) which is suitable in establishing the link between Maxwell equations of stationary media and FIEFE in Part 2 can be given as follows:

Corollary 3: For arbitrary (smooth enough) scalar/vector density fields f, g, $\vec{A}$, $\vec{B}$, $\vec{C}$ one has the maps

$$\begin{aligned} &\operatorname{div}' \vec{A}'(\vec{r}';t) + \frac{\partial f'}{\partial t}(\vec{r}';t) = g'(\vec{r}';t) \\ \rightleftarrows\ &\operatorname{div} \vec{A}(\vec{r};t) + \frac{\lozenge}{\lozenge t} f(\vec{r};t) = g(\vec{r};t) \end{aligned} \tag{4.22a}$$

$$\begin{aligned} &\operatorname{curl}' \vec{A}'(\vec{r}';t) + \frac{\partial \vec{B}'}{\partial t}(\vec{r}';t) = \vec{C}'(\vec{r}';t) \\ \rightleftarrows\ &\operatorname{curl} \vec{A}(\vec{r};t) + \frac{\lozenge}{\lozenge t} \vec{B}(\vec{r};t) = \vec{C}(\vec{r};t) \end{aligned} \tag{4.22b}$$

$$L' f'(\vec{r}';t) = g'(\vec{r}';t) \quad \rightleftarrows \quad L_\lozenge f(\vec{r};t) = g(\vec{r};t) \tag{4.22c}$$

$$L' \vec{A}'(\vec{r}';t) = \operatorname{grad}' g'(\vec{r}';t) \quad \rightleftarrows \quad L_\lozenge \vec{A}(\vec{r};t) = \operatorname{grad} g(\vec{r};t) \tag{4.22d}$$

between the two reference frames, where

$$L' = lap' - \varepsilon\mu \frac{\partial^2}{\partial t^2} - \sigma\mu \frac{\partial}{\partial t} \tag{4.22e}$$

is the stationary wave operator in L-frame, and

$$L_\lozenge = lap - \varepsilon\mu \frac{\lozenge^2}{\lozenge t^2} - \sigma\mu \frac{\lozenge}{\lozenge t} \tag{4.22f}$$

is the progressive wave operator in E-frame with ε, μ, σ being arbitrary positive constants (or constitutive parameters).

Just as in Corollary 1, the restriction of the definition of the comoving time derivative in R_3 to R_2 directly yields its expression on a regular surface by the following:

Corollary 4: (Comoving Time Derivative on a Surface)

The comoving time derivatives of a scalar field $f(\vec{r}_S;t)$ and a vector field $\vec{A}(\vec{r}_S;t)$ defined only on a two-sided, regular surface $S(t)$ traveling with a linear instantaneous velocity $\vec{v}(\vec{r}_S;t)$ is given by

$$\begin{aligned} \frac{\lozenge}{\lozenge t} f(\vec{r}_S;t) &= \frac{\partial}{\partial t} f(\vec{r}_S;t) + \operatorname{div}_S\left[\vec{v} f(\vec{r}_S;t)\right] \\ &= \frac{D}{Dt} f + f\left(\operatorname{div}_S \vec{v}\right) = \frac{\partial}{\partial t} f + \vec{v} \cdot \operatorname{grad}_S f + f\left(\operatorname{div}_S \vec{v}\right) \end{aligned} \tag{4.23a}$$

$$\begin{aligned} \frac{\lozenge}{\lozenge t} \vec{A} &= \frac{\partial}{\partial t} \vec{A} + \vec{v} \cdot \operatorname{grad}_S \vec{A} - \left(\operatorname{grad}_S \vec{v}\right) \cdot \vec{A} + \vec{A}\left(\operatorname{div}_S \vec{v}\right) \\ &= \frac{\partial}{\partial t} \vec{A} + \operatorname{div}_S\left(\vec{v}\vec{A}\right) - \vec{A} \cdot \left(\operatorname{grad}_S \vec{v}\right) \\ &= \frac{D}{Dt} \vec{A} - \vec{A} \cdot \left(\operatorname{grad}_S \vec{v}\right) + \vec{A}\left(\operatorname{div}_S \vec{v}\right) \\ &= \frac{\partial}{\partial t} \vec{A} + \vec{v} \operatorname{div}_S \vec{A} - \operatorname{curl}_S\left(\vec{v} \times \vec{A}\right) \end{aligned} \tag{4.23b}$$

Corollary 5: (Reynolds and Helmholtz Transport Theorems)

Let us consider the integral relations

$$\frac{d}{dt} \int_{\vartheta'} g'(\vec{r}';t) d\vartheta' = \int_{\vartheta'} \frac{\partial g'}{\partial t}(\vec{r}';t) d\vartheta'$$

$$\frac{d}{dt} \int_{S'} \vec{A}'(\vec{r}';t) \cdot d\vec{S}' = \int_{S'} \frac{\partial \vec{A}'}{\partial t}(\vec{r}';t) \cdot d\vec{S}'$$

for (smooth enough) scalar and vector density fields g', $\vec{A}'$ over an arbitrary volume ϑ' and a two-sided, regular surface S', respectively. Their maps into E-frame by direct substitution yield the well-known transport theorems of Reynolds and Helmholtz

$$\frac{d}{dt} \int_{\vartheta(t)} g\, d\vartheta = \int_{\vartheta(t)} \frac{\lozenge g}{\lozenge t} d\vartheta = \int_{\vartheta(t)} \frac{\partial g}{\partial t} d\vartheta + \int_{\partial\vartheta(t)} g\vec{v}\, dS \tag{4.24a}$$

$$\begin{aligned} \frac{d}{dt} \int_{S(t)} \vec{A} \cdot d\vec{S} &= \int_{S(t)} \frac{\lozenge \vec{A}}{\lozenge t} \cdot d\vec{S} \\ &= \int_{S(t)} \left(\frac{\partial \vec{A}}{\partial t} + \vec{v} \operatorname{div}_S \vec{A} \right) \cdot d\vec{S} + \int_{\partial S(t)} \left(\vec{A} \times \vec{v}\right) \cdot d\vec{S} \end{aligned} \tag{4.24b}$$

Similarly, the restriction of (4.24a) on a two-sided, regular surface in R_2 reads

$$\begin{aligned} \frac{d}{dt} \int_{S(t)} g\, dS &= \int_{S(t)} \frac{\lozenge g}{\lozenge t} dS \\ &= \int_{S(t)} \left(\frac{\partial g}{\partial t} - 2\Omega g \vec{v} \cdot \hat{n} \right) dS + \int_{\partial S(t)} g \vec{v} \cdot d\vec{C} \end{aligned} \tag{4.24c}$$

5. Properties of the Comoving Time Derivative

5.1. Certain Differential Properties

For constant quantities c, $\vec{C}$ and the scalar/vector fields $f(\vec{r};t)$, $g(\vec{r};t)$, $\vec{A}(\vec{r};t)$, $\vec{B}(\vec{r};t)$ of $C^1(R_3)$ one can observe the following properties.

Property O1:

$$\frac{\lozenge}{\lozenge t} c = c\left(\operatorname{div} \vec{v}\right) \quad , \quad \frac{\lozenge}{\lozenge t} \vec{C} = -\vec{C} \cdot \operatorname{grad} \vec{v} + \vec{C}\left(\operatorname{div} \vec{v}\right) = \operatorname{curl}\left(\vec{C} \times \vec{v}\right)$$

Property O2: $\dfrac{\lozenge}{\lozenge t}(cf) = c\dfrac{\lozenge f}{\lozenge t}$, $\dfrac{\lozenge}{\lozenge t}(c\vec{A}) = c\dfrac{\lozenge \vec{A}}{\lozenge t}$

Property O3: $\frac{\lozenge}{\lozenge t}(f \pm g) = \frac{\lozenge f}{\lozenge t} \pm \frac{\lozenge g}{\lozenge t}$, $\frac{\lozenge}{\lozenge t}(\vec{A} \pm \vec{B}) = \frac{\lozenge \vec{A}}{\lozenge t} \pm \frac{\lozenge \vec{B}}{\lozenge t}$

Property O4: $\frac{\lozenge}{\lozenge t}(fg) = \frac{\lozenge f}{\lozenge t} g + f\frac{\lozenge g}{\lozenge t} - fg(\mathrm{div}\vec{v})$

Property O5:

$$\begin{aligned}\frac{\lozenge}{\lozenge t}(f\vec{A}) &= \frac{\partial}{\partial t}(f\vec{A}) + \vec{v}\cdot\mathrm{grad}(f\vec{A}) - f\vec{A}\cdot\mathrm{grad}\vec{v} + f\vec{A}(\mathrm{div}\vec{v})\\ &= \left[\frac{\partial}{\partial t} f + \vec{v}\cdot\mathrm{grad}f + f(\mathrm{div}\vec{v})\right]\vec{A}\\ &\quad + f\left[\frac{\partial}{\partial t}\vec{A} + \vec{v}\cdot\mathrm{grad}\vec{A} - \vec{A}\cdot\mathrm{grad}\vec{v} + \vec{A}(\mathrm{div}\vec{v})\right]\\ &\quad - f\vec{A}(\mathrm{div}\vec{v})\\ &= \frac{\lozenge f}{\lozenge t}\vec{A} + f\frac{\lozenge \vec{A}}{\lozenge t} - f\vec{A}(\mathrm{div}\vec{v})\end{aligned}$$

Property O6: $\frac{\lozenge}{\lozenge t}(\vec{A}\cdot\vec{B}) = \frac{D}{Dt}(\vec{A}\cdot\vec{B}) + \vec{A}\cdot\vec{B}(\mathrm{div}\vec{v})$

Property O7:

$$\begin{aligned}\frac{\lozenge}{\lozenge t}(\vec{A}\times\vec{B}) &= \frac{D\vec{A}}{Dt}\times\vec{B} + \vec{A}\times\frac{D\vec{B}}{Dt} + \vec{A}\times\vec{B}(\mathrm{div}\vec{v}) - (\vec{A}\times\vec{B})\cdot\mathrm{grad}\vec{v}\\ &= \frac{\lozenge\vec{A}}{\lozenge t}\times\vec{B} + \vec{A}\times\frac{\lozenge\vec{B}}{\lozenge t} - \vec{A}\times\vec{B}(\mathrm{div}\vec{v}) - (\vec{A}\times\vec{B})\cdot\mathrm{grad}\vec{v}\\ &\quad + \vec{A}\times(\vec{B}\cdot\mathrm{grad}\vec{v}) + (\vec{A}\cdot\mathrm{grad}\vec{v})\times\vec{B}\end{aligned}$$

Property O8: For $n \in \mathbb{R}$

$$\begin{aligned}\frac{\lozenge}{\lozenge t} f^n &= nf^{n-1}\frac{\lozenge f}{\lozenge t} - (n-1)f^n(\mathrm{div}\vec{v}) = nf^{n-1}\frac{Df}{Dt} + f^n(\mathrm{div}\vec{v})\\ &= f^{n-1}\left[\frac{\lozenge f}{\lozenge t} + (n-1)\frac{Df}{Dt}\right]\end{aligned}$$

Property O9: $\frac{\lozenge\hat{n}}{\lozenge t} = \frac{D\hat{n}}{Dt} - \hat{n}\cdot\mathrm{grad}_S\vec{v} + \hat{n}(\mathrm{div}_S\vec{v}) = \hat{n}(\mathrm{div}_S\vec{v})$,

where $\hat{n}(t)$ is the unit normal of a regular surface $S(t)$ in arbitrary motion. This property can be verified in virtue of Property C9 and (3.2a) which requires $\hat{n}\cdot\mathrm{grad}_S\vec{v} = \vec{0}$.

Property O10: $\frac{\lozenge}{\lozenge t}(\psi\hat{n}) = \hat{n}\frac{\lozenge\psi}{\lozenge t} + \psi\frac{\lozenge\hat{n}}{\lozenge t} - \psi\hat{n}(\mathrm{div}_S\vec{v}) = \hat{n}\frac{\lozenge\psi}{\lozenge t}$, where $\hat{n}(t)$ is the unit normal of a regular surface $S(t)$ in arbitrary motion and $\psi(\vec{r}_S;t)$ is an arbitrary scalar density function.

5.2. Certain Commutative Properties

For the purpose of deriving the progressive wave equations in Part 2 we provide below three theorems investigating certain commutative properties between the comoving time and spatial (nabla) derivative operators

Theorem 7: In an arbitrarily moving material medium a density field vector $\vec{A}(\vec{r};t)$ of $C^2(R_3)$ provides the commutative properties

$$\mathrm{div}\left(\frac{\lozenge}{\lozenge t}\vec{A}\right) = \frac{\lozenge}{\lozenge t}(\mathrm{div}\vec{A}) \quad , \quad \mathrm{div}(L_{\vec{v}}\vec{A}) = L_{\vec{v}}(\mathrm{div}\vec{A}) \qquad (5.1a,b)$$

Proof: The proof requires demonstration of the equality

$$\begin{aligned}&\mathrm{div}\left[\vec{v}\cdot\mathrm{grad}\vec{A} - \vec{A}\cdot\mathrm{grad}\vec{v} + \vec{A}(\mathrm{div}\vec{v})\right]\\ &\quad = \vec{v}\cdot\mathrm{grad}(\mathrm{div}\vec{A}) + (\mathrm{div}\vec{A})(\mathrm{div}\vec{v})\end{aligned} \qquad (5.2a)$$

For this purpose we shall introduce the following tensor identities (*cf.* [19, Ch. 7])

$$\vec{A}\cdot\bar{\bar{\Phi}} = \bar{\bar{\Phi}}^{TR}\cdot\vec{A} \qquad (5.3a)$$

$$\mathrm{div}(\bar{\bar{\Phi}}\cdot\vec{A}) = (\mathrm{div}\bar{\bar{\Phi}})\cdot\vec{A} + \bar{\bar{\Phi}}:\mathrm{grad}\vec{A} \qquad (5.3b)$$

where $\vec{A}$ is a vector; $\bar{\bar{\Phi}}$ is a tensor of rank two (a dyad); ':' is the tensor inner product defined as $\vec{A}:\vec{B} = A_{ij}B_{ij}$; the superscript TR represents the transpose of the tensor when written in matrix form. From (5.3) one can write

$$\mathrm{div}(\vec{A}\cdot\bar{\bar{\Phi}}) = \mathrm{div}(\bar{\bar{\Phi}}^{TR}\cdot\vec{A}) = (\mathrm{div}\bar{\bar{\Phi}}^{TR})\cdot\vec{A} + \bar{\bar{\Phi}}^{TR}:\mathrm{grad}\vec{A} \qquad (5.3c)$$

and use this property to calculate

$$\begin{aligned}\mathrm{div}(\vec{v}\cdot\mathrm{grad}\vec{A}) &= \mathrm{div}\left[(\mathrm{grad}\vec{A})^{TR}\cdot\vec{v}\right]\\ &= \left[\mathrm{div}(\mathrm{grad}\vec{A})^{TR}\right]\cdot\vec{v} + (\mathrm{grad}\vec{A})^{TR}:\mathrm{grad}\vec{v}\\ &= \vec{v}\cdot\mathrm{grad}(\mathrm{div}\vec{A}) + \mathrm{grad}\vec{v}:(\mathrm{grad}\vec{A})^{TR}\end{aligned}$$

$$\begin{aligned}\mathrm{div}(\vec{A}\cdot\mathrm{grad}\vec{v}) &= \mathrm{div}\left[(\mathrm{grad}\vec{v})^{TR}\cdot\vec{A}\right]\\ &= \left[\mathrm{div}(\mathrm{grad}\vec{v})^{TR}\right]\cdot\vec{A} + (\mathrm{grad}\vec{v})^{TR}:\mathrm{grad}\vec{A}\\ &= \vec{A}\cdot\mathrm{grad}(\mathrm{div}\vec{v}) + \mathrm{grad}\vec{A}:(\mathrm{grad}\vec{v})^{TR}\end{aligned}$$

We also have the tensor properties

$$\mathrm{grad}\vec{v}:(\mathrm{grad}\vec{A})^{TR} = \mathrm{grad}\vec{A}:(\mathrm{grad}\vec{v})^{TR}$$

$$\mathrm{div}\left[\vec{A}(\mathrm{div}\vec{v})\right] = (\mathrm{div}\vec{A})(\mathrm{div}\vec{v}) + \left[\mathrm{grad}(\mathrm{div}\vec{v})\right]\cdot\vec{A}$$

which altogether verify the desired equality (5.2a) upon a direct substitution. A similar proof is available in the investigation in [20, Sec. 3.1] of the Maxwell - Cattaneo wave equation in heat conduction.

From Theorem 7 we observe that the commutative property between the comoving time derivative and divergence operators applies regardless of the type of motion.

Next we set to find a similar property for the divergence operators replaced with curl operator.

Theorem 8:

$$\begin{aligned}\mathrm{curl}\left(\frac{\lozenge}{\lozenge t}\vec{A}\right) &= \frac{\partial}{\partial t}\mathrm{curl}\vec{A} + \vec{v}\cdot\mathrm{grad}(\mathrm{curl}\vec{A}) + (\mathrm{curl}\vec{A})(\mathrm{div}\vec{v})\\ &\quad + \left[\mathrm{grad}(\mathrm{div}\vec{v})\right]\times\vec{A} - \vec{A}\cdot\mathrm{grad}(\mathrm{curl}\vec{v})\\ &\quad + (\mathrm{grad}\vec{v})\overset{\times}{.}(\mathrm{grad}\vec{A})^{TR} - (\mathrm{grad}\vec{A})\overset{\times}{.}(\mathrm{grad}\vec{v})^{TR}\end{aligned}$$

$$= \frac{\Diamond}{\Diamond t}\left(\text{curl}\,\vec{A}\right) - \vec{A}\cdot\text{grad}\left(\text{curl}\,\vec{v}\right) - \left(\text{curl}\vec{A}\right)\cdot\left(\text{grad}\,\vec{v}\right) + \left(\text{grad}\,\vec{v}\right)\overset{\times}{\cdot}\left(\text{grad}\,\vec{A}\right)^{TR} - \left(\text{grad}\,\vec{A}\right)\overset{\times}{\cdot}\left(\text{grad}\,\vec{v}\right)^{TR} \tag{5.4}$$

where $\overset{\times}{\cdot}$ stands for the cross-dot product in dyadic algebra defined by

$$\left(\vec{a}\vec{b}\right)\overset{\times}{\cdot}\left(\vec{c}\vec{d}\right) = \left(\vec{a}\times\vec{c}\right)\left(\vec{b}\cdot\vec{d}\right) \tag{5.5a}$$

or $$\left(\text{grad}\,\vec{a}\right)\overset{\times}{\cdot}\left(\text{grad}\,\vec{b}\right) = \left[\text{grad}\left(a_i\right)\times\text{grad}\left(b_j\right)\right]\left(\hat{x}_i\cdot\hat{x}_j\right) \tag{5.5b}$$

Proof: For our purpose let us introduce the tensor identity

$$\text{curl}\left(\overline{\overline{\Phi}}\cdot\vec{A}\right) = \left(\text{curl}\,\overline{\overline{\Phi}}\right)\cdot\vec{A} + \text{grad}\,\vec{A}\overset{\times}{\cdot}\overline{\overline{\Phi}} \tag{5.6}$$

Accordingly, one can write

$$\begin{aligned}\text{curl}\left(\vec{v}\cdot\text{grad}\,\vec{A}\right) &= \text{curl}\left[\left(\text{grad}\,\vec{A}\right)^{TR}\cdot\vec{v}\right]\\ &= \left[\text{curl}\left(\text{grad}\,\vec{A}\right)^{TR}\right]\cdot\vec{v} + \left(\text{grad}\,\vec{v}\right)\overset{\times}{\cdot}\left(\text{grad}\,\vec{A}\right)^{TR}\\ &= \vec{v}\cdot\text{grad}\left(\text{curl}\,\vec{A}\right) + \left(\text{grad}\,\vec{v}\right)\overset{\times}{\cdot}\left(\text{grad}\,\vec{A}\right)^{TR}\end{aligned}$$

$$\begin{aligned}\text{curl}\left(\vec{A}\cdot\text{grad}\,\vec{v}\right) &= \text{curl}\left[\left(\text{grad}\,\vec{v}\right)^{TR}\cdot\vec{A}\right]\\ &= \left[\text{curl}\left(\text{grad}\,\vec{v}\right)^{TR}\right]\cdot\vec{A} + \left(\text{grad}\,\vec{A}\right)\overset{\times}{\cdot}\left(\text{grad}\,\vec{v}\right)^{TR}\\ &= \vec{A}\cdot\text{grad}\left(\text{curl}\,\vec{v}\right) + \left(\text{grad}\,\vec{A}\right)\overset{\times}{\cdot}\left(\text{grad}\,\vec{v}\right)^{TR}\end{aligned}$$

and also invoke the property

$$\text{curl}\left[\vec{A}\left(\text{div}\vec{v}\right)\right] = \left(\text{curl}\vec{A}\right)\left(\text{div}\vec{v}\right) + \left[\text{grad}\left(\text{div}\vec{v}\right)\right]\times\vec{A}\ ,$$

which altogether yield the desired relation (5.4). Eq. (5.4) also signifies that there does not exist a simple commutative property between curl and the comoving time derivative operators similar in form to the case in Theorem 7 for arbitrary velocity fields. The sophisticated structure of (5.4) for arbitrary velocity fields renders it impractical in obtaining FIEFE in the most general case. However, for any medium in arbitrary Euclidean motion one can obtain the desired simple commutative property as follows:

Theorem 9: In an arbitrary material medium in Euclidean motion a density field vector $\vec{A}(\vec{r};t)$ of $C^2(R_3)$ provides the commutative properties

$$\text{curl}\left(\frac{\Diamond}{\Diamond t}\vec{A}\right) = \frac{\Diamond}{\Diamond t}\left(\text{curl}\,\vec{A}\right)\ ,\quad \text{curl}\left(L_{\vec{v}}\vec{A}\right) = L_{\vec{v}}\left(\text{curl}\,\vec{A}\right) \tag{5.7a,b}$$

Proof: Let us look into the two special cases of Euclidean motion separately.

5.3. Special Case 1: Translational Motion

In this special case described in (4.3) the comoving time derivative reduces into the classical convective derivative directly as

$$\begin{aligned}\frac{\Diamond}{\Diamond t}\vec{A} &= \left(\frac{\partial}{\partial t} + L_{\vec{v}}\right)\vec{A} = \frac{D}{Dt}\vec{A} = \left(\frac{\partial}{\partial t} + \vec{v}(t)\cdot\text{grad}\right)\vec{A}\\ &= \frac{\partial\vec{A}}{\partial t} + \text{grad}\left[\vec{v}(t)\cdot\vec{A}\right] - \vec{v}(t)\times\text{curl}\,\vec{A}\end{aligned} \tag{5.8a}$$

with

$$L_{\vec{v}}\vec{A} = \vec{v}(t)\cdot\text{grad}\,\vec{A} = \text{grad}\left[\vec{v}(t)\cdot\vec{A}\right] - \vec{v}(t)\times\text{curl}\,\vec{A} \tag{5.8b}$$

and the desired result (5.7) is seen directly upon setting the spatial derivatives of the velocity vector to zero. If we invoke the special case $\vec{v} = \vec{v}(t)$ and substitute $\text{grad}\vec{A}$ in place of $\vec{A}$ in the general vector identity

$$\text{grad}\left(\vec{v}\cdot\vec{A}\right) = \vec{v}\cdot\text{grad}\vec{A} + \vec{A}\cdot\text{grad}\,\vec{v} + \vec{A}\times\text{curl}\,\vec{v} + \vec{v}\times\text{curl}\vec{A}$$

one gets

$$\text{grad}\left[\vec{v}(t)\cdot\text{grad}\,\vec{A}\right] = \vec{v}(t)\cdot\text{grad}\left(\text{grad}\,\vec{A}\right),$$

through which one obtains an additional commutative property

$$\text{grad}\left(\frac{\Diamond}{\Diamond t}\vec{A}\right) = \frac{\Diamond}{\Diamond t}\left(\text{grad}\,\vec{A}\right)\ ,\quad \text{grad}\left(L_{\vec{v}}\vec{A}\right) = L_{\vec{v}}\left(\text{grad}\,\vec{A}\right) \tag{5.8c,d}$$

required in deriving the potential wave operators.

5.4. Special Case 2: Rotational Motion

In this special case described in (4.4) the velocity vector has the properties

$$\text{grad}\,\vec{v} = \omega(t)\left(\hat{\rho}\hat{\phi} - \hat{\phi}\hat{\rho}\right),\ \text{curl}\,\vec{v} = 2\omega(t)\hat{z}\ ,\ \text{div}\,\vec{v} = 0\ ,\ \vec{a} = \dot{\omega}(t)\rho\hat{\phi}$$

$$\left(\text{grad}\,\vec{v}\right)^{TR} = -\text{grad}\,\vec{v}\ ,\quad \text{grad}\left(\text{div}\vec{v}\right) = \vec{0}\ ,\quad \text{grad}\left(\text{curl}\,\vec{v}\right) = \vec{0}$$

where the dot over angular frequency (and over any quantity for the rest of the investigation) indicates ordinary time derivative. Then for general field quantities in the form $f = f(\rho,\phi,z;t)$,

$$\vec{A}(\rho,\phi,z;t) = \hat{\rho}A_\rho(\rho,\phi,z;t) + \hat{\phi}A_\phi(\rho,\phi,z;t) + \hat{z}A_z(\rho,\phi,z;t)\,,$$

one obtains

$$L_{\vec{v}}f = \vec{v}\cdot\text{grad}\,f = \omega(t)\frac{\partial f}{\partial\phi} \tag{5.9a}$$

$$\begin{aligned}L_{\vec{v}}\vec{A} &= \vec{v}\cdot\text{grad}\,\vec{A} - \vec{A}\cdot\text{grad}\,\vec{v} = \omega(t)\left(\frac{\partial\vec{A}}{\partial\phi} - \hat{z}\times\vec{A}\right)\\ &= \omega(t)\left(\hat{\rho}\frac{\partial A_\rho}{\partial\phi} + \hat{\phi}\frac{\partial A_\phi}{\partial\phi} + \hat{z}\frac{\partial A_z}{\partial\phi}\right)\end{aligned} \tag{5.9b}$$

$$\begin{aligned}\frac{D}{Dt}f &= \left(\frac{\partial}{\partial t} + \vec{v}\cdot\text{grad}\right)f\\ &= \left(\frac{\partial}{\partial t} + \omega(t)\frac{\partial}{\partial\phi}\right)f = \left(\frac{\partial}{\partial t} + L_{\vec{v}}\right)f = \frac{\Diamond}{\Diamond t}f\end{aligned} \tag{5.9c}$$

$$\frac{D}{Dt}\vec{A} = \left(\frac{\partial}{\partial t} + \vec{\mathrm{v}}\cdot\mathrm{grad}\right)\vec{A} = \left(\frac{\partial}{\partial t} + \omega(t)\frac{\partial}{\partial\phi}\right)\vec{A}$$
$$= \left(\frac{\partial}{\partial t} + L_{\vec{\mathrm{v}}} + \omega(t)\hat{z}\times\right)\vec{A} = \left(\frac{\lozenge}{\lozenge t} + \omega(t)\hat{z}\times\right)\vec{A} \tag{5.9d}$$

$$\frac{\lozenge}{\lozenge t}(\mathrm{grad}\, f) = \left(\frac{D}{Dt} - \omega(t)\hat{z}\times\right)(\mathrm{grad}\, f) = \mathrm{grad}\left(\frac{D}{Dt}f\right) \tag{5.9e}$$

$$(1/\omega)(\mathrm{grad}\,\vec{\mathrm{v}})\underset{\cdot}{\times}(\mathrm{grad}\,\vec{A})^{TR}$$
$$= -\hat{\rho}\frac{\partial A_z}{\partial\rho} - \frac{\hat{\phi}}{\rho}\frac{\partial A_z}{\partial\phi} + \hat{z}\left(\frac{\partial A_\rho}{\partial\rho} + \frac{1}{\rho}\left(A_\rho + \frac{\partial A_\phi}{\partial\phi}\right)\right)$$

$$(1/\omega)(\mathrm{grad}\,\vec{A})\underset{\cdot}{\times}(\mathrm{grad}\,\vec{\mathrm{v}})^{TR} = -(1/\omega)(\mathrm{grad}\,\vec{A})\underset{\cdot}{\times}(\mathrm{grad}\,\vec{\mathrm{v}})$$
$$= -\hat{\rho}\frac{\partial A_\rho}{\partial z} - \hat{\phi}\frac{\partial A_\phi}{\partial z} + \hat{z}\left[\frac{\partial A_\rho}{\partial\rho} + \frac{1}{\rho}\left(A_\rho + \frac{\partial A_\phi}{\partial\phi}\right)\right]$$

$$(\mathrm{grad}\,\vec{\mathrm{v}})\underset{\cdot}{\times}(\mathrm{grad}\,\vec{A})^{TR} - (\mathrm{grad}\,\vec{A})\underset{\cdot}{\times}(\mathrm{grad}\,\vec{\mathrm{v}})^{TR} = -\omega(t)\hat{z}\times\mathrm{curl}\,\vec{A}$$

$$(\mathrm{curl}\,\vec{A})\cdot(\mathrm{grad}\,\vec{\mathrm{v}}) = (\mathrm{grad}\,\vec{\mathrm{v}})^{TR}\cdot(\mathrm{curl}\,\vec{A}) = -(\mathrm{grad}\,\vec{\mathrm{v}})\cdot(\mathrm{curl}\,\vec{A})$$

$$\frac{\lozenge}{\lozenge t}(\mathrm{grad}\,\vec{A}) = \mathrm{grad}\left(\left(\frac{\lozenge}{\lozenge t} + \omega(t)\hat{z}\times\right)\vec{A}\right)$$
$$= \mathrm{grad}\left(\frac{\lozenge}{\lozenge t}\vec{A}\right) - \omega(t)(\mathrm{grad}\,\vec{A})\times\hat{z} \tag{5.9f}$$
$$= \mathrm{grad}\left(\frac{D}{Dt}\vec{A}\right)$$

$$L_{\vec{\mathrm{v}}}(\mathrm{grad}\vec{A}) = \mathrm{grad}\left[(L_{\vec{\mathrm{v}}} + \omega(t)\hat{z}\times)\vec{A}\right]$$
$$= \mathrm{grad}(L_{\vec{\mathrm{v}}}\vec{A}) - \omega(t)(\mathrm{grad}\vec{A})\times\hat{z} \tag{5.9g}$$
$$= \mathrm{grad}\left(\left(\frac{D}{Dt} - \frac{\partial}{\partial t}\right)\vec{A}\right) = \omega(t)\mathrm{grad}\left(\frac{\partial\vec{A}}{\partial\phi}\right)$$

Placing the resultant expressions into (5.4) we obtain the desired result (5.7), which completes the proof of Theorem 9:

$$\mathrm{curl}\left(\frac{\lozenge}{\lozenge t}\vec{A}\right) = \frac{\lozenge}{\lozenge t}(\mathrm{curl}\,\vec{A}) - (\mathrm{curl}\,\vec{A})\cdot(\mathrm{grad}\,\vec{\mathrm{v}})$$
$$+ (\mathrm{grad}\,\vec{\mathrm{v}})\underset{\cdot}{\times}(\mathrm{grad}\,\vec{A})^{TR} - (\mathrm{grad}\,\vec{A})\underset{\cdot}{\times}(\mathrm{grad}\,\vec{\mathrm{v}})^{TR}$$
$$= \frac{\lozenge}{\lozenge t}(\mathrm{curl}\,\vec{A}) + \left[\mathrm{grad}\,\vec{\mathrm{v}}\cdot - \omega(t)\hat{z}\times\right](\mathrm{curl}\,\vec{A})$$
$$= \left(\frac{D}{Dt} - \omega(t)\hat{z}\times\right)(\mathrm{curl}\vec{A}) = \frac{\lozenge}{\lozenge t}(\mathrm{curl}\vec{A})$$

Certain of the important results obtained for the two special cases of Euclidean motion are depicted in Table 1. An investigation of these properties for media in non-Euclidean motion, especially involving radial motion with a velocity field in the general form $\vec{\mathrm{v}}(\vec{r};t) = \alpha(\vec{r};t)\vec{r}$ characterizing expansion or contraction mechanisms with specific applications in electromagnetic theory, are left as the subject of a separate work.

Type of Motion	Translational	Rotational
Coordinate Maps	$x_i = x'_i + \int_{-\infty}^{t} \mathrm{v}^i(\xi)\, d\xi$	$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} \cos\phi(t) & \sin\phi(t) \\ -\sin\phi(t) & \cos\phi(t) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $x'_3 = x_3$
Description of Velocity	$\vec{\mathrm{v}}(\vec{r};t) = \vec{\mathrm{v}}(t)$	$\vec{\mathrm{v}}(\rho,\phi,z;t) = \omega(t)\rho\hat{\phi}$
Differential Properties of Velocity	$\mathrm{div}\,\vec{\mathrm{v}} = 0$ $\mathrm{curl}\,\vec{\mathrm{v}} = \vec{0}$	$\mathrm{div}\,\vec{\mathrm{v}} = 0$ $\mathrm{curl}\,\vec{\mathrm{v}} = 2\omega(t)\hat{z}$
Velocity Gradient	$\bar{\bar{L}} = \mathrm{grad}\,\vec{\mathrm{v}} = \bar{\bar{0}}$	$\bar{\bar{L}} = \mathrm{grad}\,\vec{\mathrm{v}} = \omega(t)(\hat{\rho}\hat{\phi} - \hat{\phi}\hat{\rho})$
Deformation Gradient	$\bar{\bar{F}} = \left\Vert\frac{\partial x_i}{\partial x'_j}\right\Vert = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \bar{\bar{I}}$	$\bar{\bar{F}} = \left\Vert\frac{\partial x_i}{\partial x'_j}\right\Vert = \begin{bmatrix} \cos\phi(t) & -\sin\phi(t) \\ \sin\phi(t) & \cos\phi(t) \end{bmatrix} = \bar{\bar{Q}}^{TR}(t)$
Jacobian of Deformation Gradient	$J = \det\left\Vert\frac{\partial x_i}{\partial x'_j}\right\Vert = 1$	$J = \det\left\Vert\frac{\partial x_i}{\partial x'_j}\right\Vert = 1$
Convective Derivative	$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{\mathrm{v}}(t)\cdot\mathrm{grad}$	$\frac{D}{Dt} = \frac{\partial}{\partial t} + \omega(t)\frac{\partial}{\partial\phi}$
Comoving Time Derivatives	$\frac{\lozenge}{\lozenge t}f = \frac{D}{Dt}f$ $\frac{\lozenge}{\lozenge t}\vec{A} = \frac{D}{Dt}\vec{A}$	$\frac{\lozenge}{\lozenge t}f = \frac{D}{Dt}f$ $\frac{\lozenge}{\lozenge t}\vec{A} = \left(\frac{D}{Dt} - \omega(t)\hat{z}\times\right)\vec{A}$
Certain Differential Properties	$\frac{\lozenge}{\lozenge t}(fg) = \frac{D}{Dt}(fg)$	$\frac{\lozenge}{\lozenge t}(fg) = \frac{D}{Dt}(fg)$

	$\frac{\Diamond}{\Diamond t}(f\vec{A})=\frac{D}{Dt}(f\vec{A})$ $\frac{\Diamond}{\Diamond t}(\vec{A}\cdot\vec{B})=\frac{D}{Dt}(\vec{A}\cdot\vec{B})$ $\frac{\Diamond}{\Diamond t}(\vec{A}\times\vec{B})=\frac{D}{Dt}(\vec{A}\times\vec{B})$	$\frac{\Diamond}{\Diamond t}(f\vec{A})=\frac{D}{Dt}(f\vec{A})$ $\frac{\Diamond}{\Diamond t}(\vec{A}\cdot\vec{B})=\frac{D}{Dt}(\vec{A}\cdot\vec{B})$ $\frac{\Diamond}{\Diamond t}(\vec{A}\times\vec{B})=\frac{\Diamond\vec{A}}{\Diamond t}\times\vec{B}+\vec{A}\times\frac{\Diamond\vec{B}}{\Diamond t}$
Commutative Properties	$\frac{\Diamond}{\Diamond t}(\text{grad}\,\vec{A})=\text{grad}\left(\frac{\Diamond}{\Diamond t}\vec{A}\right)=\text{grad}\left(\frac{D}{Dt}\vec{A}\right)$ $\frac{\Diamond}{\Diamond t}(\text{curl}\,\vec{A})=\text{curl}\left(\frac{\Diamond}{\Diamond t}\vec{A}\right)=\text{curl}\left(\frac{D}{Dt}\vec{A}\right)$	$\frac{\Diamond}{\Diamond t}(\text{grad}\,\vec{A})=\text{grad}\left(\frac{D}{Dt}\vec{A}\right)$ $\frac{\Diamond}{\Diamond t}(\text{curl}\,\vec{A})=\text{curl}\left(\frac{\Diamond}{\Diamond t}\vec{A}\right)$

Table 1. Certain analytical results for two special types of Euclidean motion

PART 2: THE AXIOMATIC STRUCTURE

6. Maxwell Equations of Stationary Media

We consider a medium with arbitrary electromagnetic properties in arbitrary motion with respect to an observer considered at rest in E-frame as depicted in Fig. 4.

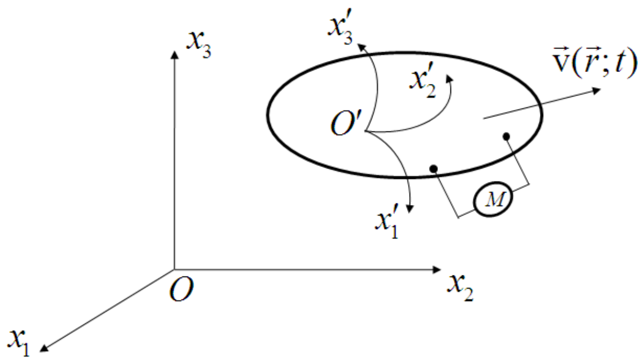

Fig. 4. E- and L-frames of an electromagnetic medium in arbitrary motion

In L-frame denoted with primes the medium is considered locally at rest ('stationary'). By definition, convective currents $\vec{J}'_V(\vec{r}';t)$ occur when a material medium is in motion and therefore are avoided in L-configuration, by which one infers that the free currents in stationary media constitute only conduction currents, namely

$$\vec{J}'_f(\vec{r}';t)\equiv\vec{J}'_C(\vec{r}';t)\,.$$

Accordingly, we introduce the following postulate:

Postulate 2: Macroscopic electromagnetic phenomena of stationary media are governed by the Maxwell equations

$$\text{curl}'\,\vec{E}'(\vec{r}';t)+\frac{\partial}{\partial t}\vec{B}'(\vec{r}';t)=\vec{0} \qquad (6.1a)$$

$$\text{curl}'\,\vec{H}'(\vec{r}';t)-\frac{\partial}{\partial t}\vec{D}'(\vec{r}';t)=\vec{J}'_C(\vec{r}';t) \qquad (6.1b)$$

$$\text{div}'\,\vec{D}'(\vec{r}';t)=\rho'_f(\vec{r}';t) \qquad (6.1c)$$

$$\text{div}'\,\vec{B}'(\vec{r}';t)=0 \qquad (6.1d)$$

or equivalently, the integral set

$$\oint_{\partial S'}\vec{E}'\cdot d\vec{c}'+\frac{d}{dt}\int_{S'}\vec{B}'\cdot d\vec{S}'=\vec{0} \qquad (6.2a)$$

$$\oint_{\partial S'}\vec{H}'\cdot d\vec{c}'+\frac{d}{dt}\int_{S'}\vec{D}'\cdot d\vec{S}'=\int_{S'}\vec{J}'_C\cdot d\vec{S}' \qquad (6.2b)$$

$$\oint_{\partial\vartheta'}\vec{D}'\cdot d\vec{S}'=\int_{\vartheta'}\rho'_f\,d\vartheta' \qquad (6.2c)$$

$$\oint_{\partial\vartheta'}\vec{B}'\cdot d\vec{S}'=0 \qquad (6.2d)$$

where also we involve the closed form constitutive relations

$$\vec{D}'=\vec{f}_d\left(\vec{E}';\vec{H}'\right)=\varepsilon_0\vec{E}'+\vec{P}^{e\prime} \qquad (6.3a)$$

$$\vec{B}'=\vec{f}_b\left(\vec{E}';\vec{H}'\right)=\mu_0\vec{H}'+\vec{P}^{m\prime} \qquad (6.3b)$$

$$\vec{J}'_C=\vec{f}_C\left(\vec{E}';\vec{H}'\right) \qquad (6.3c)$$

Here ϑ' and S' are arbitrary regular volume and surface regions which are stationary in L-frame and all primed field quantities are described and measured by an L-observer, i.e., by an ideal measurement device mounted on any measurement point in the medium.

When the Maxwell equations are considered as the fundamental laws of stationary media, then the continuity relation

$$\text{div}'\,\vec{J}'_C(\vec{r}';t)+\frac{\partial}{\partial t}\rho'_f(\vec{r}';t)=0 \qquad (6.4a)$$

or equivalently, $$\oint_{\partial\vartheta'}\vec{J}'_C\cdot d\vec{S}'+\frac{d}{dt}\int_{\vartheta'}\rho'_f\cdot d\vartheta'=0 \qquad (6.4b)$$

follows as a corollary.

The Lorentz potentials in L-frame are given by

$$\vec{B}'(\vec{r}';t)=\text{curl}'\,\vec{A}(\vec{r}';t) \qquad (6.5a)$$

$$\vec{E}'(\vec{r}';t)=-\frac{\partial}{\partial t}\vec{A}'(\vec{r}';t)-\text{grad}'\,V'(\vec{r}';t) \qquad (6.5b)$$

The Poynting theorem in the L-frame in point form is given by

$$\text{div}\vec{P}'+\vec{E}'\cdot\vec{J}_d^{e\prime}+\vec{H}'\cdot\vec{J}_d^{m\prime}+\vec{E}'\cdot\vec{J}'_C=0 \qquad (6.6a)$$

where

$$\vec{P}'(\vec{r}';t)=\vec{E}'(\vec{r}';t)\times\vec{H}'(\vec{r}';t) \qquad (6.6b)$$

is the usual Poynting vector and

$$\vec{J}_d^{e\prime}(\vec{r}';t)=\frac{\partial}{\partial t}\vec{D}'(\vec{r}';t)\quad,\quad\vec{J}_d^{m\prime}(\vec{r}';t)=\frac{\partial}{\partial t}\vec{B}'(\vec{r}';t) \qquad (6.6c,d)$$

stand for the electric and magnetic displacement current densities regardless of the constitutive parameters of the medium involved.

The integral form of Poynting theorem (6.6a-d) in ϑ' is written as

$$P'_{in}(\vec{r}';t) = P_d^{e'}(\vec{r}';t) + P_d^{m'}(\vec{r}';t) + P'_C(\vec{r}';t) \tag{6.6e}$$

where

$$P'_{in} = -\oint_{\partial\vartheta'} \vec{P}' \cdot d\vec{S}' \tag{6.6f}$$

$$P_d^{e'} = \int_{\vartheta'} \vec{E}' \cdot \vec{J}_d^{e'}\, d\vartheta' = \int_{\vartheta'} \vec{E}' \cdot \frac{\partial \vec{D}'}{\partial t}\, d\vartheta' \tag{6.6g}$$

$$P_d^{m'} = \int_{\vartheta'} \vec{H}' \cdot \vec{J}_d^{m'}\, d\vartheta' = \int_{\vartheta'} \vec{H}' \cdot \frac{\partial \vec{B}'}{\partial t}\, d\vartheta' \tag{6.6h}$$

$$P'_C = \int_{\vartheta'} \vec{E}' \cdot \vec{J}'_C\, d\vartheta' \tag{6.6i}$$

and can be interpreted as follows:

The total electromagnetic power P'_{in} entering (or pumped by external sources to) an arbitrary stationary material medium ϑ' is equal to the sum of

1. the total electrical power $P_d^{e'}$ stored in that medium;
2. the total magnetic power $P_d^{m'}$ stored in that medium;
3. the total electrical power P'_C dissipated as heat in that medium.

In a simple medium described by the constitutive relations

$$\vec{D}' = \varepsilon\vec{E}' \quad , \quad \vec{B}' = \mu\vec{H}' \quad , \quad \vec{J}'_C = \sigma\vec{E}' \tag{6.7a-c}$$

the well known wave equations

$$L'\vec{E}'(\vec{r}';t) = (1/\varepsilon)\,\mathrm{grad}'\,\rho'_f(\vec{r}';t) \tag{6.8a}$$

$$L'\vec{H}'(\vec{r}';t) = \vec{0} \tag{6.8b}$$

$$L'\vec{A}'(\vec{r}';t) = \vec{0} \tag{6.8c}$$

$$L'V'(\vec{r}';t) = -(1/\varepsilon)\,\rho'_f(\vec{r}';t) \tag{6.8d}$$

and the Lorentz gauge relation

$$\mathrm{div}\,\vec{A}'(\vec{r}';t) + \varepsilon\mu\frac{\partial}{\partial t}V'(\vec{r}';t) + \sigma\mu V'(\vec{r}';t) = 0 \tag{6.8e}$$

The Lorentz force law in L-frame is an additional (external) postulate to Maxwell's field theory given by the following:

Postulate 3: The mechanical force acting on a material point charge at rest in Maxwell's field theory of stationary media is described by the Lorentz force law, which we express for the force volume density field as

$$\vec{f}'(\vec{r}';t) = \frac{d\vec{F}'}{d\vartheta'}(\vec{r}';t) = \rho'_f(\vec{r}';t)\vec{E}'(\vec{r}';t) + \vec{J}'_C(\vec{r}';t) \times \vec{B}'(\vec{r}';t) \tag{6.9}$$

The Lorentz force law is our unique bridge connecting the disciplines of electromagnetism and mechanics.

We shall also outline the special cases of electrostatic and magnetostatic field equations of stationary media in L-frame as

$$\mathrm{curl}'\,\vec{E}'(\vec{r}') = \vec{0} \tag{6.10a}$$

$$\mathrm{div}'\,\vec{D}'(\vec{r}') = \rho'_f(\vec{r}') \tag{6.10b}$$

$$\vec{E}'(\vec{r}') = -\mathrm{grad}'\,V'(\vec{r}') \tag{6.10c}$$

$$lap'\,V'(\vec{r}') = -(1/\varepsilon)\,\rho'_f(\vec{r}') \quad \text{(in a simple medium)} \tag{6.10d}$$

$$\vec{f}'(\vec{r}') = \frac{d\vec{F}'}{d\vartheta'}(\vec{r}') = \rho'_f(\vec{r}')\vec{E}'(\vec{r}') \tag{6.10e}$$

and

$$\mathrm{curl}'\,\vec{H}'(\vec{r}') = \vec{J}'_C(\vec{r}') \tag{6.11a}$$

$$\mathrm{div}'\,\vec{B}'(\vec{r}') = 0 \tag{6.11b}$$

$$\mathrm{div}'\,\vec{J}'_C(\vec{r}') = 0 \tag{6.11c}$$

$$\vec{B}'(\vec{r}') = \mathrm{curl}'\,\vec{A}(\vec{r}') \tag{6.11d}$$

$$\mathrm{div}'\vec{A}'(\vec{r}') = 0 \quad \text{(Coulomb gauge)} \tag{6.11e}$$

$$\mathrm{lap}'\vec{A}'(\vec{r}') = \mu\vec{J}'_C(\vec{r}') \quad \text{(in a simple medium)} \tag{6.11f}$$

7. Frame Indifferent Electromagnetic Field Equations

We describe the conjecture of the principle of frame indifference onto electromagnetism through the following postulate:

Postulate 4: The laws of macroscopic electromagnetism are frame indifferent.

This is another way to saying that the observer in E-frame is in full agreement with

- the nature of the physical quantities
- the structural forms of physical laws, and
- any measurement result taken

in L-frame in virtue of the mathematical rules presented in Part 1. We claim Postulates 1 to 4 to be sufficient in constructing a material description of macroscopic electromagnetism in *continuous* media and they bring about the following corollaries.

Corollary 6: All primed field quantities in L-frame preserve their forms (including the functional space they belong), nature and measurement values for any observer in unprimed E-frame through the relations

$$\begin{aligned} q'(\vec{r}';t) &= q(\vec{r};t) \quad , \quad \rho'_f(\vec{r}';t) = \rho_f(\vec{r};t) \\ \vec{E}'(\vec{r}';t) &= \bar{\bar{Q}}(t)\cdot\vec{E}(\vec{r};t) \quad , \quad \vec{D}'(\vec{r}';t) = \bar{\bar{Q}}(t)\cdot\vec{D}(\vec{r};t) \\ \vec{H}'(\vec{r}';t) &= \bar{\bar{Q}}(t)\cdot\vec{H}(\vec{r};t) \quad , \quad \vec{B}'(\vec{r}';t) = \bar{\bar{Q}}(t)\cdot\vec{B}(\vec{r};t) \\ \vec{J}'_C(\vec{r}';t) &= \bar{\bar{Q}}(t)\cdot\vec{J}_C(\vec{r};t) \quad , \quad \vec{F}'(\vec{r}';t) = \bar{\bar{Q}}(t)\cdot\vec{F}(\vec{r};t) \end{aligned} \tag{7.1}$$

Corollary 7: The frame indifferent spatial gradient (nabla) and comoving time derivatives are enrolled to preserve the frame indifference of electromagnetic field equations. Accordingly, in virtue of Corollary 3, the images of Maxwell equations in L-frame introduce the FIEFE in E-frame in the form

$$\operatorname{curl}\vec{E}(\vec{r};t)+\frac{\Diamond}{\Diamond t}\vec{B}(\vec{r};t)=\vec{0} \qquad (7.2a)$$

$$\operatorname{curl}\vec{H}(\vec{r};t)-\frac{\Diamond}{\Diamond t}\vec{D}(\vec{r};t)=\vec{J}_C(\vec{r};t) \qquad (7.2b)$$

$$\operatorname{div}\vec{D}(\vec{r};t)=\rho_f(\vec{r};t) \qquad (7.2c)$$

$$\operatorname{div}\vec{B}(\vec{r};t)=0 \qquad (7.2d)$$

$$\operatorname{div}\vec{J}_C(\vec{r};t)+\frac{\Diamond}{\Diamond t}\rho_f(\vec{r};t)=0 \qquad (7.2e)$$

and equivalently, the integral set

$$\oint_{\partial S(t)}\vec{E}\cdot d\vec{c}+\frac{d}{dt}\int_{S(t)}\vec{B}\cdot d\vec{S}=\vec{0} \qquad (7.3a)$$

$$\oint_{\partial S(t)}\vec{H}\cdot d\vec{c}-\frac{d}{dt}\int_{S(t)}\vec{D}\cdot d\vec{S}=\int_{S(t)}\vec{J}_C\cdot d\vec{S} \qquad (7.3b)$$

$$\oint_{\partial\vartheta(t)}\vec{D}\cdot d\vec{S}=\int_{\vartheta(t)}\rho_f\,d\vartheta \qquad (7.3c)$$

$$\oint_{\partial\vartheta(t)}\vec{B}\cdot d\vec{S}=0 \qquad (7.3d)$$

$$\oint_{\partial\vartheta(t)}\vec{J}_C\cdot d\vec{S}+\frac{d}{dt}\int_{\vartheta(t)}\rho_f\,d\vartheta=0 \qquad (7.3e)$$

and the constitutive relations in closed form

$$\vec{D}=\vec{f}_d\left(\vec{E};\vec{H}\right)=\varepsilon_0\vec{E}+\vec{P}^e \quad , \quad \vec{B}=\vec{f}_b\left(\vec{E};\vec{H}\right)=\mu_0\vec{H}+\vec{P}^m \qquad (7.4a,b)$$

$$\vec{J}_C=\vec{f}_C\left(\vec{E};\vec{H}\right) \qquad (7.4c)$$

Here $\vartheta(t)$ and $S(t)$ are the images of ϑ' and S' in E-frame. In E-frame we describe the convective current and the total free current by

$$\vec{J}_V=\vec{\mathrm{v}}\rho_f \quad , \quad \vec{J}_f=\vec{J}_C+\vec{J}_V \qquad (7.4d,e)$$

through which the continuity relations can be shaped as

$$\operatorname{div}\vec{J}_f+\frac{\partial}{\partial t}\rho_f=0 \qquad (7.5a)$$

$$\oint_{\partial\vartheta(t)}\vec{J}_f\cdot d\vec{S}+\int_{\vartheta(t)}\frac{\partial}{\partial t}\rho_f\,d\vartheta=0 \qquad (7.5b)$$

Also the electromotive and magnetomotive forces (emf & mmf) measured in L-frame over S' can be expressed in E-frame directly by the maps

$$\begin{aligned}\mathrm{emf}(t)&=\oint_{\partial S'}\vec{E}'\cdot d\vec{c}'=\oint_{\partial S(t)}\vec{E}\cdot d\vec{c}=-\frac{d}{dt}\int_{S(t)}\vec{B}\cdot d\vec{S}\\&=-\int_{S(t)}\frac{\Diamond}{\Diamond t}\vec{B}\cdot d\vec{S}=-\int_{S(t)}\frac{\partial}{\partial t}\vec{B}\cdot d\vec{S}+\oint_{\partial S(t)}\left(\vec{\mathrm{v}}\times\vec{B}\right)\cdot d\vec{c}\end{aligned} \qquad (7.6a)$$

$$\begin{aligned}\mathrm{mmf}(t)&=\oint_{\partial S'}\vec{H}'\cdot d\vec{c}'=\oint_{\partial S(t)}\vec{H}\cdot d\vec{c}\\&=\frac{d}{dt}\int_{S(t)}\vec{D}\cdot d\vec{S}+\int_{S(t)}\vec{J}_C\cdot d\vec{S}=\int_{S(t)}\frac{\Diamond}{\Diamond t}\vec{D}\cdot d\vec{S}+\int_{S(t)}\vec{J}_C\cdot d\vec{S}\\&=\int_{S(t)}\frac{\partial}{\partial t}\vec{D}\cdot d\vec{S}+\int_{S(t)}\left(\vec{J}_C+\vec{\mathrm{v}}\rho_f\right)\cdot d\vec{S}-\oint_{\partial S(t)}\left(\vec{\mathrm{v}}\times\vec{D}\right)\cdot d\vec{c}\\&=\int_{S(t)}\frac{\partial}{\partial t}\vec{D}\cdot d\vec{S}+\int_{S(t)}\vec{J}_f\cdot d\vec{S}-\oint_{\partial S(t)}\left(\vec{\mathrm{v}}\times\vec{D}\right)\cdot d\vec{c}\end{aligned} \qquad (7.6b)$$

Equations (7.2a,b) can also be written in the familiar form

$$\operatorname{curl}\left(\vec{E}-\vec{\mathrm{v}}\times\vec{B}\right)+\frac{\partial}{\partial t}\vec{B}=\vec{0} \qquad (7.7a)$$

$$\operatorname{curl}\left(\vec{H}+\vec{\mathrm{v}}\times\vec{D}\right)-\frac{\partial}{\partial t}\vec{D}=\vec{J}_f \qquad (7.7b)$$

upon inserting

$$\begin{aligned}\frac{\Diamond}{\Diamond t}\vec{D}&=\frac{\partial}{\partial t}\vec{D}+\vec{\mathrm{v}}\operatorname{div}\vec{D}-\operatorname{curl}\left(\vec{\mathrm{v}}\times\vec{D}\right)\\&=\frac{\partial}{\partial t}\vec{D}+\vec{\mathrm{v}}\rho_f-\operatorname{curl}\left(\vec{\mathrm{v}}\times\vec{D}\right)\\&=\frac{\partial}{\partial t}\vec{D}+\vec{J}_V-\operatorname{curl}\left(\vec{\mathrm{v}}\times\vec{D}\right)\end{aligned} \qquad (7.8a)$$

$$\begin{aligned}\frac{\Diamond}{\Diamond t}\vec{B}&=\frac{\partial}{\partial t}\vec{B}+\vec{\mathrm{v}}\operatorname{div}\vec{B}-\operatorname{curl}\left(\vec{\mathrm{v}}\times\vec{B}\right)\\&=\frac{\partial}{\partial t}\vec{B}-\operatorname{curl}\left(\vec{\mathrm{v}}\times\vec{B}\right)\end{aligned} \qquad (7.8b)$$

Regarding the Lorentz potentials, from (7.2d) and (7.7a) one can directly write

$$\vec{B}(\vec{r};t)=\operatorname{curl}\vec{A}(\vec{r};t) \qquad (7.9a)$$

$$\begin{aligned}\vec{E}(\vec{r};t)&=\vec{\mathrm{v}}(\vec{r};t)\times\vec{B}(\vec{r};t)-\frac{\partial}{\partial t}\vec{A}(\vec{r};t)-\operatorname{grad}V(\vec{r};t)\\&=\vec{\mathrm{v}}(\vec{r};t)\times\operatorname{curl}\vec{A}(\vec{r};t)-\frac{\partial}{\partial t}\vec{A}(\vec{r};t)-\operatorname{grad}V(\vec{r};t)\\&=\operatorname{grad}\left(\vec{\mathrm{v}}\cdot\vec{A}\right)-\frac{D}{Dt}\vec{A}-\vec{A}\cdot\operatorname{grad}\vec{\mathrm{v}}\\&\quad-\vec{A}\times\operatorname{curl}\vec{\mathrm{v}}-\operatorname{grad}V\end{aligned} \qquad (7.9b)$$

regardless of the constitutive parameters of the medium involved. In his book [21, Ch.5] T.E. Phipps defines the last two terms at the r.h.s. of (7.7b) as 'the Maxwell $\vec{E}$ - field'

$$\vec{E}_{\max}(\vec{r};t)=-\frac{\partial}{\partial t}\vec{A}(\vec{r};t)-\operatorname{grad}V(\vec{r};t) \qquad (7.10)$$

based on its structural similarity with (6.5b).

The introduction of the comoving time derivative requires us to describe the electric/magnetic displacement current density of the medium in E-frame as

$$\vec{J}_d^e(\vec{r};t)=\frac{\Diamond}{\Diamond t}\vec{D}(\vec{r};t)=\frac{\partial}{\partial t}\vec{D}(\vec{r};t)+L_{\vec{\mathrm{v}}}\vec{D}(\vec{r};t) \qquad (7.11a)$$

$$\vec{J}_d^m(\vec{r};t)=\frac{\Diamond}{\Diamond t}\vec{B}(\vec{r};t)=\frac{\partial}{\partial t}\vec{B}(\vec{r};t)+L_{\vec{\mathrm{v}}}\vec{B}(\vec{r};t) \qquad (7.11b)$$

where

$$L_{\vec{v}}\vec{D}(\vec{r};t)=\vec{J}_V-\mathrm{curl}\left(\vec{v}\times\vec{D}\right) \tag{7.11c}$$

$$L_{\vec{v}}\vec{B}(\vec{r};t)=-\mathrm{curl}\left(\vec{v}\times\vec{B}\right) \tag{7.11d}$$

Then the Poynting theorem of the medium in motion can be written in the following forms

$$\mathrm{div}\vec{P}+\vec{E}\cdot\vec{J}_d^e+\vec{H}\cdot\vec{J}_d^m+\vec{E}\cdot\vec{J}_f=0 \tag{7.12a}$$

$$\mathrm{div}\vec{P}+\vec{E}\cdot\frac{\lozenge}{\lozenge t}\vec{D}+\vec{H}\cdot\frac{\lozenge}{\lozenge t}\vec{B}+\vec{E}\cdot\vec{J}_C+\vec{E}\cdot\vec{J}_V=0 \tag{7.12b}$$

$$\begin{aligned}\mathrm{div}\vec{P}+\vec{E}\cdot\frac{\partial}{\partial t}\vec{D}+\vec{E}\cdot L_{\vec{v}}\vec{D}+\vec{H}\cdot\frac{\partial}{\partial t}\vec{B}\\ +\vec{H}\cdot L_{\vec{v}}\vec{B}+\vec{E}\cdot\vec{J}_C+\vec{E}\cdot\vec{J}_V=0\end{aligned} \tag{7.12c}$$

while the Poynting vector in E-frame is defined in the usual form

$$\vec{P}(\vec{r};t)=\vec{E}(\vec{r};t)\times\vec{H}(\vec{r};t)\ . \tag{7.12d}$$

The integral form of Poynting theorem in E-frame is expressed by mapping (6.6e-i) as

$$P_{in}(\vec{r};t)=P_d^e(\vec{r};t)+P_d^m(\vec{r};t)+P_C(\vec{r};t) \tag{7.12e}$$

where

$$P_{in}=-\oint_{\partial\vartheta(t)}\vec{P}\cdot d\vec{S} \tag{7.12f}$$

$$\begin{aligned}P_d^e&=\int_{\vartheta(t)}\vec{E}\cdot\vec{J}_d^e\,d\vartheta=\int_{\vartheta(t)}\vec{E}\cdot\frac{\lozenge}{\lozenge t}\vec{D}\,d\vartheta\\ &=\int_{\vartheta(t)}\vec{E}\cdot\frac{\partial}{\partial t}\vec{D}\,d\vartheta+\int_{\vartheta(t)}\vec{E}\cdot L_{\vec{v}}\vec{D}\,d\vartheta\end{aligned} \tag{7.12g}$$

$$=\int_{\vartheta(t)}\vec{E}\cdot\frac{\partial}{\partial t}\vec{D}\,d\vartheta+\int_{\vartheta(t)}\vec{E}_{Max}\cdot\vec{J}_V\,d\vartheta-\int_{\vartheta(t)}\vec{E}\cdot\mathrm{curl}\left(\vec{v}\times\vec{D}\right)d\vartheta$$

$$\begin{aligned}P_d^m&=\int_{\vartheta(t)}\vec{H}\cdot\vec{J}_d^m\,d\vartheta=\int_{\vartheta(t)}\vec{H}\cdot\frac{\lozenge}{\lozenge t}\vec{B}\,d\vartheta\\ &=\int_{\vartheta(t)}\vec{H}\cdot\frac{\partial}{\partial t}\vec{B}\,d\vartheta+\int_{\vartheta(t)}\vec{H}\cdot L_{\vec{v}}\vec{B}\,d\vartheta\\ &=\int_{\vartheta(t)}\vec{H}\cdot\frac{\partial}{\partial t}\vec{B}\,d\vartheta-\int_{\vartheta(t)}\vec{H}\cdot\mathrm{curl}\left(\vec{v}\times\vec{B}\right)d\vartheta\end{aligned} \tag{7.12h}$$

$$P_C=\int_{\vartheta(t)}\vec{E}\cdot\vec{J}_C\,d\vartheta \tag{7.12i}$$

In a simple medium with constitutive parameters (6.7a,b) the first integrals at the r.h.s. of (7.12g,h) can also be written as

$$\begin{aligned}\int_{\vartheta(t)}\vec{E}\cdot\frac{\partial}{\partial t}\vec{D}\,d\vartheta&=\int_{\vartheta(t)}\varepsilon\vec{E}\cdot\frac{\partial}{\partial t}\vec{E}\,d\vartheta=\int_{\vartheta(t)}\frac{\partial}{\partial t}\left(\tfrac{1}{2}\varepsilon\vec{E}^2\right)d\vartheta\\ &=\frac{d}{dt}\int_{\vartheta(t)}\tfrac{1}{2}\varepsilon\vec{E}^2\,d\vartheta-\oint_{\partial\vartheta(t)}\tfrac{1}{2}\varepsilon\vec{E}^2\,\vec{v}\cdot d\vec{S}\end{aligned} \tag{7.12j}$$

$$\begin{aligned}\int_{\vartheta(t)}\vec{H}\cdot\frac{\partial}{\partial t}\vec{B}\,d\vartheta&=\int_{\vartheta(t)}\mu\vec{H}\cdot\frac{\partial}{\partial t}\vec{H}\,d\vartheta=\int_{\vartheta(t)}\frac{\partial}{\partial t}\left(\tfrac{1}{2}\mu\vec{H}^2\right)d\vartheta\\ &=\frac{d}{dt}\int_{\vartheta(t)}\tfrac{1}{2}\mu\vec{H}^2\,d\vartheta-\oint_{\partial\vartheta(t)}\tfrac{1}{2}\mu\vec{H}^2\,\vec{v}\cdot d\vec{S}\end{aligned} \tag{7.12k}$$

In the final steps of calculation in (7.12j,k) we employed the scalar Reynolds theorem.

The Lorentz force law in E-frame takes the form

$$\vec{f}(\vec{r};t)=\frac{d\vec{F}}{d\vartheta}(\vec{r};t)=\rho_f(\vec{r};t)\vec{E}(\vec{r};t)+\vec{J}_C(\vec{r};t)\times\vec{B}(\vec{r};t)\ . \tag{7.13}$$

Substituting (7.9b) and (7.10) into (7.13) gives

$$\vec{f}(\vec{r};t)=\frac{d\vec{F}}{d\vartheta}(\vec{r};t)=\rho_f(\vec{r};t)\vec{E}_{Max}(\vec{r};t)+\vec{J}_f(\vec{r};t)\times\vec{B}(\vec{r};t)\ . \tag{7.14}$$

The resultant expression (7.14), which is generally postulated directly as the Lorentz force law, is actually the image of the Lorentz force law (6.9) of stationary media, regardless of the choice of Lorentz gauge. Detailed discussion around (7.14) can be found at [21 ,Ch.5].

The image in E-frame of the electrostatic and magnetostatic field equations of stationary media in (6.10) and (6.11) can be written respectively as

$$\mathrm{curl}\,\vec{E}(\vec{r};t)=\vec{0} \tag{7.15a}$$

$$\frac{\lozenge}{\lozenge t}\vec{D}(\vec{r};t)=\vec{0} \tag{7.15b}$$

$$\mathrm{div}\,\vec{D}(\vec{r};t)=\rho_f(\vec{r};t) \tag{7.15c}$$

$$\vec{E}(\vec{r};t)=-\mathrm{grad}\,V(\vec{r};t) \tag{7.15d}$$

$$\frac{\lozenge}{\lozenge t}\rho_f(\vec{r};t)=0 \tag{7.15e}$$

$$\vec{f}(\vec{r};t)=\frac{d\vec{F}}{d\vartheta}(\vec{r};t)=\rho_f(\vec{r};t)\vec{E}(\vec{r};t) \tag{7.15f}$$

and

$$\frac{\lozenge}{\lozenge t}\vec{B}(\vec{r};t)=\vec{0} \tag{7.16a}$$

$$\mathrm{curl}\,\vec{H}(\vec{r};t)=\vec{J}_C(\vec{r};t) \tag{7.16b}$$

$$\mathrm{div}\,\vec{B}(\vec{r};t)=0 \tag{7.16c}$$

$$\mathrm{div}\,\vec{J}_C(\vec{r};t)=0 \tag{7.16d}$$

$$\vec{B}(\vec{r};t)=\mathrm{curl}\,\vec{A}(\vec{r};t) \tag{7.16e}$$

$$\mathrm{div}\vec{A}(\vec{r};t)=0\quad\text{(Coulomb gauge)} \tag{7.16f}$$

$$\vec{f}(\vec{r};t)=\frac{d\vec{F}}{d\vartheta}(\vec{r};t)=\vec{J}_C(\vec{r};t)\times\vec{B}(\vec{r};t) \tag{7.16g}$$

It should be emphasized that while the electrostatic field quantities in L-frame are observed as time dependent in E-frame as in (7.15), this does not imply a presence of an additional magnetic field. What happens is that the field lines follow the arbitrary motion of the source as a whole, *without* any deformation in shape. Therefore it should not be mixed with any type of radiation mechanism specific to time varying sources where the field lines change their shape in L-frame. In that regard FIEFT put is very clearly that "*stationary (time independent) sources with arbitrary velocity do not radiate*". Similar considerations hold for a magnetostatic medium in (7.16).

Next we shall seek the wave equations and Lorentz potentials in simple media for the two special cases of Euclidean motion summarized in Table 1.

8. Frame Indifferent Wave Equations and Lorentz Potentials in Simple Media

8.1. Special Case 1: Translational Motion

In this case it is sufficient to replace the partial time derivative $\frac{\partial}{\partial t}$ in the Maxwell equations in L-frame with convective derivative as $\frac{\lozenge}{\lozenge t} = \frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{\mathrm{v}}(t)\cdot \mathrm{grad} = \frac{\partial}{\partial t} + L_{\vec{\mathrm{v}}}$, which shapes (7.2a,b,e) into

$$\mathrm{curl}\,\vec{E}(\vec{r};t) + \frac{D}{Dt}\vec{B}(\vec{r};t) = \vec{0} \qquad (8.1a)$$

$$\mathrm{curl}\,\vec{H}(\vec{r};t) - \frac{D}{Dt}\vec{D}(\vec{r};t) = \vec{J}_C(\vec{r};t) \qquad (8.1b)$$

$$\mathrm{div}\,\vec{J}_C(\vec{r};t) + \frac{D}{Dt}\rho_f(\vec{r};t) = 0 \qquad (8.1c)$$

In virtue of Lemma 2 the wave equations for fields and Lorentz potentials can be adapted directly from the well-known results (6.8) in stationary case as

$$L_D\vec{E}(\vec{r};t) = (1/\varepsilon)\,\mathrm{grad}\,\rho_f(\vec{r};t) \qquad (8.2a)$$

$$L_D\vec{H}(\vec{r};t) = \vec{0} \qquad (8.2b)$$

$$L_D\vec{A}(\vec{r};t) = \vec{0} \qquad (8.2c)$$

$$L_D V(\vec{r};t) = -(1/\varepsilon)\,\rho_f(\vec{r};t) \qquad (8.2d)$$

$$\mathrm{div}\,\vec{A}(\vec{r};t) + \varepsilon\mu\frac{D}{Dt}V(\vec{r};t) + \sigma\mu V(\vec{r};t) = 0 \qquad (8.2e)$$

$$\begin{aligned}\vec{E} &= \vec{\mathrm{v}}(t)\times\mathrm{curl}\,\vec{A} - \frac{\partial}{\partial t}\vec{A} - \mathrm{grad}\,V \\ &= \mathrm{grad}\left(\vec{\mathrm{v}}(t)\cdot\vec{A}\right) - \frac{D}{Dt}\vec{A} - \mathrm{grad}\,V\end{aligned} \qquad (8.2f)$$

where we define the reduced progressive (or convective) wave operator

$$L_D = \mathrm{lap} - \varepsilon\mu\frac{D^2}{Dt^2} - \sigma\mu\frac{D}{Dt}. \qquad (8.2g)$$

To understand the nature of the vector convective wave operator (8.2g) let us consider the special case of R_1 where

$$\vec{\mathrm{v}}(t) = \hat{x}_1\mathrm{v}(t) \quad , \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathrm{v}(t)\frac{\partial}{\partial x_1}. \qquad (8.3a,b)$$

In this case each field component satisfies the scalar convective wave operator

$$\begin{aligned}L_D = &\left[1-\varepsilon\mu\mathrm{v}^2(t)\right]\frac{\partial^2}{\partial x_1^2} + 2\varepsilon\mu\mathrm{v}(t)\frac{\partial^2}{\partial x_1\partial t} - \varepsilon\mu\frac{\partial^2}{\partial t^2} \\ &+\left[\sigma\mu\mathrm{v}(t) - \varepsilon\mu\mathrm{a}(t)\right]\frac{\partial}{\partial x_1} - \sigma\mu\frac{\partial}{\partial t}\end{aligned} \qquad (8.3c)$$

where $\mathrm{a}(t) = \dot{v}(t)$ is the acceleration.

The discriminant of the partial differential operator in (8.3c) reads

$$\Delta = 4\varepsilon^2\mu^2\mathrm{v}^2(t) - 4\left[1-\varepsilon\mu\mathrm{v}^2(t)\right](-\varepsilon\mu) = 4\varepsilon\mu > 0 \qquad (8.3d)$$

which altogether provide the following evidences:

1. The discriminant values of the wave operators in L- and E-frames are the same (invariant). Therefore the vector operator L_D in E-frame is of hyperbolic type regardless of the instantaneous value of the velocity of the material points.
2. $1-\varepsilon\mu\mathrm{v}^2(t) = 0$, which also describes the speed of light in a simple medium, is a critical value for the velocity of material points in a simple medium, for which the wave propagation phenomenon *breaks down*.

It should be noticed that the investigation so far does not introduce any upper limit for the speeds of material points; meaning that $1-\varepsilon\mu\mathrm{v}^2(t)$ can also take negative values in (8.3c), which might address a possibility of speeds of material points faster than the speed of light in the same simple medium.

The field equations (8.1-8.2) are known as Hertz equations since they indicate the farthest point Hertz was able to reach in his theoretical studies and publish in his 1890 paper (see [22, Ch.XIV]) at age 32. Soon afterwards he got a serious infection and passed away in 1894 without a chance to pursue his research.

8.2. Special Case 2: Rotational Motion

In this case the wave equations and Lorentz potentials for the fields in a simple medium can be written directly as

$$L_\lozenge\vec{E}(\vec{r};t) = (1/\varepsilon)\,\mathrm{grad}\,\rho_f(\vec{r};t) \qquad (8.4a)$$

$$L_\lozenge\vec{H}(\vec{r};t) = \vec{0} \qquad (8.4b)$$

$$L_\lozenge\vec{A}(\vec{r};t) = \vec{0} \qquad (8.4c)$$

$$L_\lozenge V(\vec{r};t) = L_D V(\vec{r};t) = -(1/\varepsilon)\,\rho_f(\vec{r};t) \qquad (8.4d)$$

$$\mathrm{div}\,\vec{A}(\vec{r};t) + \varepsilon\mu\frac{D}{Dt}V(\vec{r};t) + \sigma\mu V(\vec{r};t) = 0 \qquad (8.4e)$$

$$\begin{aligned}\vec{E} &= \vec{\mathrm{v}}\times\mathrm{curl}\,\vec{A} - \frac{\partial}{\partial t}\vec{A} - \mathrm{grad}\,V \\ &= \mathrm{grad}\left(\vec{\mathrm{v}}\cdot\vec{A}\right) - \frac{\lozenge}{\lozenge t}\vec{A} - \mathrm{grad}\,V\end{aligned} \qquad (8.4f)$$

In the special case of R_1 where $\frac{\partial}{\partial\rho}\equiv 0$, $\frac{\partial}{\partial z}\equiv 0$, each field component satisfies the reduced form

$$L_\Diamond = \left(\frac{1}{\rho^2} - \varepsilon\mu\omega^2(t)\right)\frac{\partial^2}{\partial\phi^2} + 2\varepsilon\mu\omega(t)\frac{\partial^2}{\partial\phi\partial t} - \varepsilon\mu\frac{\partial^2}{\partial t^2} \qquad (8.5a)$$
$$+\text{lower order terms}$$

of the progressive wave operator for which the discriminant reads

$$\Delta = 4\varepsilon\mu/\rho^2 > 0 \qquad (8.5b)$$

and therefore similar physical arguments as for translational motion hold.

PART 3: BOUNDARY VALUE PROBLEMS

9. General Formulation

The linear structure of electromagnetic field equations require that the field expressions in E-frame in any scenario of moving bodies should be obtainable through the images of the end results obtained in the corresponding Maxwell's theory of stationary media (in other words, in L-frame). Therefore we can solve a scattering problem from an isolated moving body formally by frame hopping following the steps below:

1. Map the incoming field from E- to L-frame
2. Solve the scattered field from the associated boundary value problem in L-frame
3. Map the scattered field from L- to E-frame

In complementing the boundary value problem in L-frame, the corresponding spatial/temporal jump and edge conditions are obtained from the distributional investigation of the field equations (7.2)-(7.16), which constitute the final postulate.

Postulate 5: The Maxwell equations of stationary media are valid in the sense of Schwartz-Sobolev distributions.

Although the distributional results of Maxwell equations were derived and utilized much earlier in literature, to the best of the author's knowledge, this fact was introduced as a postulate first by İdemen [23] in 1973. Along with other types of complementary conditions such as radiation condition, periodicity, *etc.*, we can consider the description of any boundary value problem is formally completed.

In what follows let us consider the scenario in Fig.5 where, according to an E-observer, the incident electromagnetic wave with fields $\left(\vec{E}_{inc}(\vec{r};t), \vec{H}_{inc}(\vec{r};t)\right)$ and sources $\left(\rho_{Tx}(\vec{r};t), \vec{J}_{Tx}(\vec{r};t)\right)$ generated by a transmitter assumed stationary for E-observer in medium *I* is impinging on an object occupying a region D and in arbitrary relative motion with velocity $\vec{\mathrm{v}}(\vec{r};t)$

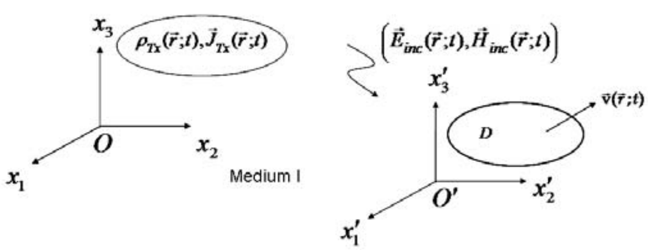

Fig. 5. An illustration of a scattering problem

9.1. The Incoming Wave

In E-frame $Ox_1x_2x_3$ the incident fields (with argument $(\vec{r};t)$) satisfy the Maxwell equations of stationary media

$$\operatorname{curl}\vec{E}_{inc} + \frac{\partial}{\partial t}\vec{B}_{inc} = \vec{0} \ , \ \operatorname{curl}\vec{H}_{inc} - \frac{\partial}{\partial t}\vec{D}_{inc} = \vec{J}_{Tx} \qquad (9.1a,b)$$

$$\operatorname{div}\vec{D}_{inc} = \rho_{Tx} \quad , \quad \operatorname{div}\vec{B}_{inc} = 0 \ . \qquad (9.1c,d)$$

Let us assume the medium I simple and lossless with constitutive parameters (ε,μ). Then the incident fields in E-frame satisfy the stationary wave and Helmholtz equations

$$\left(lap - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\vec{E}_{inc}\\ \vec{H}_{inc}\end{pmatrix} = \begin{pmatrix}(1/\varepsilon)\,grad\,\rho_{Tx}\\ \vec{0}\end{pmatrix} \qquad (9.2a)$$

$$\left(lap + k^2\right)\begin{pmatrix}\vec{E}_{inc}\\ \vec{H}_{inc}\end{pmatrix} = \begin{pmatrix}(1/\varepsilon)\,grad\,\rho_{Tx}\\ \vec{0}\end{pmatrix} \qquad (9.2b)$$

where $c = 1/\sqrt{\mu\varepsilon}$ is the phase velocity and $k = \omega\sqrt{\mu\varepsilon} = 2\pi/\lambda$ is the wave number with time dependence taken as $\exp(-i\omega t)$. For an observer in L-frame $Ox_1'x_2'x_3'$ the object is stationary and the surrounding medium I is in relative motion with a velocity $\vec{\mathrm{v}}'(\vec{r}';t)$. Accordingly, in L-frame the incident fields (with argument $(\vec{r}';t)$) satisfy the frame indifferent field and wave equations

$$\operatorname{curl}'\vec{E}_{inc}' + \frac{\Diamond'}{\Diamond' t}\vec{B}_{inc}' = \vec{0} \ , \ \operatorname{curl}'\vec{H}_{inc}' - \frac{\Diamond'}{\Diamond' t}\vec{D}_{inc}' = \vec{J}_{Tx}' \qquad (9.3a,b)$$

$$\operatorname{div}'\vec{D}_{inc}' = \rho_{Tx}' \ , \ \operatorname{div}'\vec{B}_{inc}' = 0 \qquad (9.3c,d)$$

$$\left(lap' - \frac{1}{c^2}\frac{\Diamond'^2}{\Diamond' t^2}\right)\begin{pmatrix}\vec{E}_{inc}'\\ \vec{H}_{inc}'\end{pmatrix} = \begin{pmatrix}(1/\varepsilon)\,grad'\rho_{Tx}'\\ \vec{0}\end{pmatrix} \qquad (9.4a)$$

$$\left(lap' + k^2\right)\begin{pmatrix}\vec{E}_{inc}'\\ \vec{H}_{inc}'\end{pmatrix} = \begin{pmatrix}(1/\varepsilon)\,grad'\rho_{Tx}'\\ \vec{0}\end{pmatrix} \qquad (9.4b)$$

Here the comoving time derivative of a vector $\vec{A}_{inc}'$ is given by

$$\begin{aligned}\frac{\Diamond'}{\Diamond' t}\vec{A}' &= \frac{D'}{D't}\vec{A}' - \vec{A}'\cdot grad'\vec{\mathrm{v}}' + \vec{A}'\left(div'\vec{\mathrm{v}}'\right)\\ &= \frac{\partial}{\partial t}\vec{A}' + \vec{\mathrm{v}}'\cdot grad'\vec{A}' - \vec{A}'\cdot grad'\vec{\mathrm{v}}' + \vec{A}'\left(div'\vec{\mathrm{v}}'\right)\end{aligned} \qquad (9.5)$$

9.2. The Scattered Wave

Let us express the total field in space in E- and L-frames respectively as

$$\left(\vec{E}_{tot},\vec{H}_{tot}\right) = \begin{cases}\left(\vec{E}_{inc},\vec{H}_{inc}\right) + \left(\vec{E}_{sc},\vec{H}_{sc}\right), \text{in medium I}\\ \left(\vec{E}_d,\vec{H}_d\right), \text{in region D}\end{cases}$$

and $$\left(\vec{E}_{tot}',\vec{H}_{tot}'\right) = \begin{cases}\left(\vec{E}_{inc}',\vec{H}_{inc}'\right) + \left(\vec{E}_{sc}',\vec{H}_{sc}'\right), \text{in medium I}\\ \left(\vec{E}_d',\vec{H}_d'\right), \text{in region D}\end{cases}.$$

In L-frame of the scattered field, i.e. according to an observer traveling with the scattered field, the ambient source-free medium I is in motion with linear velocity $-\vec{\mathrm{v}}'(\vec{r}';t)$. Accordingly, the scattered fields in medium I satisfy the frame indifferent field and wave equations

$$\text{curl}'\vec{E}'_{sc}+\frac{\overline{\Diamond}'}{\overline{\Diamond}'t}\vec{B}'_{sc}=\vec{0}\quad,\quad \text{curl}'\vec{H}'_{sc}-\frac{\overline{\Diamond}'}{\overline{\Diamond}'t}\vec{D}'_{sc}=\vec{0} \tag{9.6a,b}$$

$$\text{div}'\vec{D}'_{sc}=0\quad,\quad \text{div}'\vec{B}'_{sc}=0 \tag{9.6c,d}$$

$$\left(lap'-\frac{1}{c^2}\frac{\overline{\Diamond}'^2}{\overline{\Diamond}'t^2}\right)\begin{pmatrix}\vec{E}'_{sc}\\ \vec{H}'_{sc}\end{pmatrix}=\vec{0}\quad,\quad \left(lap'+k^2\right)\begin{pmatrix}\vec{E}'_{sc}\\ \vec{H}'_{sc}\end{pmatrix}=\vec{0} \tag{9.7a,b}$$

where the accompanying comoving time derivative of a vector $\vec{A}'_{sc}$ is defined as

$$\begin{aligned}\frac{\overline{\Diamond}'}{\overline{\Diamond}'t}\vec{A}'_{sc}&=\frac{D'}{D't}\vec{A}'_{sc}+\vec{A}'_{sc}\cdot grad'\vec{\text{v}}'-\vec{A}'_{sc}\left(div'\vec{\text{v}}'\right)\\ &=\frac{\partial}{\partial t}\vec{A}'_{sc}-\vec{\text{v}}'\cdot grad'\vec{A}'_{sc}+\vec{A}'_{sc}\cdot grad'\vec{\text{v}}'-\vec{A}'_{sc}\left(div'\vec{\text{v}}'\right)\end{aligned} \tag{9.8}$$

9.3. Total Field inside the Moving Object

In L-frame of the region D with fields $\left(\vec{E}'_d,\vec{H}'_d\right)$ and sources $\left(\rho'_d,\vec{J}'_d\right)$, the region is stationary since the ambient medium I is observed as source-free. Therefore in region D the field equations of stationary media

$$\text{curl}'\vec{E}'_d+\frac{\partial}{\partial t}\vec{B}'_d=\vec{0}\quad,\quad \text{curl}'\vec{H}'_d-\frac{\partial}{\partial t}\vec{D}'_d=\vec{J}'_d \tag{9.9a,b}$$

$$\text{div}'\vec{D}'_d=\rho'_d\quad,\quad \text{div}'\vec{B}'_d=0 \tag{9.9c,d}$$

are satisfied. When the region D simple with constitutive parameters $\left(\varepsilon_d,\mu_d,\sigma_d\right)$, (9.9) yield the stationary wave equations

$$\left(lap'-\frac{1}{c_d^2}\frac{\partial^2}{\partial t^2}-\sigma_d\mu_d\frac{\partial}{\partial t}\right)\begin{pmatrix}\vec{E}'_d\\ \vec{H}'_d\end{pmatrix}=\begin{pmatrix}\left(1/\varepsilon_d\right)grad'\rho'_d\\ \vec{0}\end{pmatrix} \tag{9.10a}$$

$$\left(lap'+k_d^2\right)\begin{pmatrix}\vec{E}'_d\\ \vec{H}'_d\end{pmatrix}=\begin{pmatrix}\left(1/\varepsilon_d\right)grad'\rho'_d\\ \vec{0}\end{pmatrix} \tag{9.10b}$$

with $c_d=1\big/\sqrt{\mu_d\varepsilon_d}$, $k_d^2=\omega_{inc}^2\varepsilon_d\mu_d+i\omega_{inc}\sigma_d\mu_d$. For E-observer the field and wave equations (9.9), (9.10) read

$$\text{curl}\,\vec{E}_d+\frac{\Diamond}{\Diamond t}\vec{B}_d=\vec{0}\quad,\quad \text{curl}\,\vec{H}_d-\frac{\Diamond}{\Diamond t}\vec{D}_d=\vec{J}_d \tag{9.11a,b}$$

$$\text{div}\,\vec{D}_d=\rho_d\quad,\quad \text{div}\,\vec{B}_d=0 \tag{9.11c,d}$$

$$\left(lap-\frac{1}{c_d^2}\frac{\Diamond^2}{\Diamond t^2}-\sigma_d\mu_d\frac{\Diamond}{\Diamond t}\right)\begin{pmatrix}\vec{E}_d\\ \vec{H}_d\end{pmatrix}=\begin{pmatrix}\left(1/\varepsilon_d\right)grad\rho_d\\ \vec{0}\end{pmatrix} \tag{9.12}$$

where the accompanying comoving time derivative of a vector $\vec{A}_d$ is defined as

$$\begin{aligned}\frac{\Diamond}{\Diamond t}\vec{A}_d&=\frac{D}{Dt}\vec{A}_d-\vec{A}_d\cdot\text{grad}\vec{\upsilon}+\vec{A}_d\left(\text{div}\vec{\upsilon}\right)\\ &=\frac{\partial}{\partial t}\vec{A}_d+\vec{\upsilon}\cdot\text{grad}\,\vec{A}_d-\vec{A}_d\cdot\text{grad}\vec{\upsilon}+\vec{A}_d\left(\text{div}\vec{\upsilon}\right)\end{aligned} \tag{9.13}$$

9.4. Boundary Relations on the Moving Object

On the enclosure $S=\partial D$ of the moving medium, which is assumed a simple interface that may support surface charges and currents $\left(\rho'_S(\vec{r}'_S;t),\vec{J}_S(\vec{r}'_S;t)\right)$, the distributional form of stationary field (Maxwell) equations in L- frame read

$$\hat{n}'\times\left[\vec{E}'_{inc}(\vec{r}'_S;t)+\vec{E}'_{sc}(\vec{r}'_S;t)\right]=\hat{n}'\times\vec{E}'_d(\vec{r}'_S;t) \tag{9.14a}$$

$$\hat{n}'\times\left[\vec{H}'_{inc}(\vec{r}'_S;t)+\vec{H}'_{sc}(\vec{r}'_S;t)\right]-\hat{n}'\times\vec{H}'_d(\vec{r}'_S;t)=\vec{J}'_S(\vec{r}'_S;t) \tag{9.14b}$$

$$\hat{n}'\cdot\left[\vec{D}'_{inc}(\vec{r}'_S;t)+\vec{D}'_{sc}(\vec{r}'_S;t)\right]-\hat{n}'\cdot\vec{D}'_d(\vec{r}'_S;t)=\rho'_S(\vec{r}'_S;t) \tag{9.14c}$$

$$\hat{n}'\cdot\left[\vec{B}'_{inc}(\vec{r}'_S;t)+\vec{B}'_{sc}(\vec{r}'_S;t)\right]=\hat{n}'\cdot\vec{B}'_d(\vec{r}'_S;t) \tag{9.14d}$$

Along with constitutive relations and radiation, edge, tip, periodicity, etc. type conditions complementing the boundary relations, the associated boundary value problem can be solved uniquely to yield the L-fields $\left(\vec{E}'_{sc},\vec{H}'_{sc}\right)$ and $\left(\vec{E}'_d,\vec{H}'_d\right)$, whose images also yield the E-fields $\left(\vec{E}_{sc},\vec{H}_{sc}\right)$ and $\left(\vec{E}_d,\vec{H}_d\right)$.

In the following sections we shall investigate three canonical problems of practical interest to demonstrate the predictions of FIEFT.

10. TM Plane Wave Scattering by a Moving Dielectric Half Space

In this section we shall investigate the scattering of uniform homogeneous TM plane waves by a lossless dielectric half space for two different modes of motion of practical interest. Common to both cases is the expression of the incident wave in region $x_1<0$, which propagates along $\hat{n}_{inc}=(\cos\alpha,\sin\alpha)$ direction in (x_1,x_2) plane with fields represented by

$$\begin{aligned}\vec{H}_{inc}(\vec{r};t)&=\hat{x}_3 f(\hat{n}_{inc}\cdot\vec{r}-ct)\\ &=\hat{x}_3 f(x_1\cos\alpha+x_2\sin\alpha-ct)\end{aligned} \tag{10.1a}$$

$$\vec{E}_{inc}(\vec{r};t)=Z\vec{H}_{inc}(\vec{r};t)\times\hat{n}_{inc} \tag{10.1b}$$

where $Z=\sqrt{\mu/\varepsilon}$ stands for the characteristic impedance. For the special case of monochromatic source the incident magnetic field is assumed to have the general form

$$\vec{H}_{inc}(\vec{r};t)=\hat{x}_3 g(k\hat{n}_{inc}\cdot\vec{r}-\omega_{inc}t) \tag{10.1c}$$

with angular frequency $\omega_{inc}=2\pi f_{inc}$ and phase velocity $\vec{\text{v}}_{\text{P}inc}=\hat{x}_1 c=\hat{x}_1\,\omega_{inc}/k=\hat{x}_1\lambda f_{inc}$, while (10.1b) still holds. Without losing generality, let us assume the half-space $x_1>0$ lossless with constitutive parameters $\left(\varepsilon_d,\mu_d\right)$, characteristic impedance $Z_d=\sqrt{\mu_d/\varepsilon_d}$, wave number $k_d=\omega_{inc}\sqrt{\varepsilon_d\mu_d}$ and refractivity defined by $n=c/c_d=k_d/k=\sqrt{\varepsilon_d\mu_d}\big/\sqrt{\varepsilon\mu}$.

In the first of the two special cases below we carry out the investigation for general time harmonic and monochromatic waves simultaneously.

10.1. Case I: Uniform Motion

For E-observer we assume the half-space $x_1>0$ in uniform rectilinear motion with velocity $\vec{\text{v}}=\pm G\hat{x}_1$, $G=const>0$, while for L-observer the same medium is stationary and it is half-space $x'_1<0$ (medium I) moving with linear velocity $\vec{\text{v}}'=\mp G\hat{x}'_1$. Incorporating the coordinate transformations $x_1 = x_1' \pm Gt$, $x_2 = x_2'$, $x_3 = x_3'$; $\hat{x}_i = \hat{x}_i'$, $i = 1,2,3$, the incoming fields in L-frame can be given as

$$\vec{H}_{\text{inc}}'(\vec{r}';t) = \hat{x}_3' f(\hat{n}_{\text{inc}}' \cdot \vec{r}' - c_{\text{inc}}' t) = \hat{x}_3' g(k\hat{n}_{\text{inc}}' \cdot \vec{r}' - \omega_{\text{inc}}' t) \quad (10.2a)$$

$$\vec{E}_{\text{inc}}'(\vec{r}';t) = Z\vec{H}_{\text{inc}}'(\vec{r}';t) \times \hat{n}_{\text{inc}}' \quad (10.2b)$$

with

$$\hat{n}_{\text{inc}}' \cdot \vec{r}' = x_1' \cos\alpha + x_2' \sin\alpha \ ,$$

$$c_{\text{inc}}' = c \mp G\cos\alpha = c(1 \mp \beta\cos\alpha) \ , \ \omega_{\text{inc}}' = \omega_{\text{inc}}(1 \mp \beta\cos\alpha) \ ,$$

$$f_{\text{inc}}' = f_{\text{inc}}(1 \mp \beta\cos\alpha) \ .$$

We observe $\beta < \cos\alpha$ as a physical limit on G for the realization of scattering phenomenon. The scattered field and the total field in region D can be given by

$$\vec{H}_{sc}'(\vec{r}';t) = \hat{x}_3' R_{TM} f(\hat{n}_{sc}' \cdot \vec{r}' - c_{sc}' t) = \hat{x}_3' R_{TM} g(k\hat{n}_{sc}' \cdot \vec{r}' - \omega_{sc}' t) \quad (10.3a)$$

$$\vec{E}_{sc}'(\vec{r}';t) = Z\vec{H}_{sc}'(\vec{r}';t) \times \hat{n}_{sc}' \quad (10.3b)$$

$$\vec{H}_d'(\vec{r}';t) = \hat{x}_3' T_{TM} f(\hat{n}_d' \cdot \vec{r}' - c_d' t) = \hat{x}_3' T_{TM} g(k_d \hat{n}_d' \cdot \vec{r}' - \omega_d' t) \quad (10.4a)$$

$$\vec{E}_d'(\vec{r}';t) = Z_d \vec{H}_d'(\vec{r}';t) \times \hat{n}_d' \quad (10.4b)$$

$$\hat{n}_{sc}' \cdot \vec{r}' = -x_1' \cos\alpha_{sc} + x_2' \sin\alpha_{sc} \ , \ \hat{n}_d' \cdot \vec{r}' = x_1' \cos\alpha_d + x_2' \sin\alpha_d \ .$$

The unknown quantities c_{sc}', α_{sc}, c_d', α_d, R_{TM}, T_{TM} are solved from the boundary value problem

$$\begin{cases} \left(lap' - \frac{1}{c^2}\frac{\bar{\Diamond}'^2}{\bar{\Diamond}' t^2}\right)\vec{H}_{sc}'(x_1',x_2',x_3';t) = \vec{0}, \text{in medium I} \\ \left(lap' - \frac{1}{c_d^2}\frac{\partial^2}{\partial t^2}\right)\vec{H}_d'(x_1',x_2',x_3';t) = \vec{0}, \text{in region D} \\ \quad \vec{H}_{inc}'(0,x_2',x_3';t) + \vec{H}_{sc}'(0,x_2',x_3';t) \\ \quad = \vec{H}_d'(0,x_2',x_3';t), \ \ \forall x_2',x_3',t \\ Z\left[\vec{H}_{\text{inc}}'(0,x_2',x_3';t) \times \hat{n}_{\text{inc}}' + \vec{H}_{sc}'(0,x_2',x_3';t) \times \hat{n}_{sc}'\right] \\ = Z_d \vec{H}_d'(0,x_2',x_3';t) \times \hat{n}_d', \ \ \forall x_2',x_3',t \\ \quad \text{Radiation Conditions as } x_1' \to \pm\infty \end{cases} \quad (10.5a\text{-}e)$$

The wave equation (10.5a) in medium I, namely,

$$\left(lap' - \frac{1}{c^2}\frac{\bar{\Diamond}'^2}{\bar{\Diamond}' t^2}\right)\vec{H}_{sc}' = \hat{x}_3'\left(lap' - \frac{1}{c^2}\left(\frac{\partial}{\partial t} \pm G\frac{\partial}{\partial x_1'}\right)^2\right) f(\hat{n}_{sc}' \cdot \vec{r}' - c_{sc}' t)$$

$$= \hat{x}_3'\left(1 - \frac{1}{c^2}\left(-c_{sc}' \mp G\cos\alpha_{sc}\right)^2\right) f = \vec{0}$$

require $c_{sc}' = c(1 \mp \beta\cos\alpha_{sc})$, $\omega_{sc}' = \omega_{\text{inc}}(1 \mp \beta\cos\alpha_{sc})$, while (10.5b) in region D require $c_d' = c_d$ and $\omega_d' = \omega_{\text{inc}}$. Boundary conditions (10.5c,d) require

$$\begin{aligned} &f(x_2'\sin\alpha - c_{\text{inc}}' t) + R_{TM} f(x_2'\sin\alpha_{sc} - c_{sc}' t) \\ &\quad = T_{TM} f(x_2'\sin\alpha_d - c_d' t), \quad \forall x_2', t \end{aligned} \quad (10.6a)$$

$$\begin{aligned} &Z\left[\cos\alpha\, f(x_2'\sin\alpha - c_{\text{inc}}' t) - \cos\alpha_{sc} R_{TM} f(x_2'\sin\alpha_{sc} - c_{sc}' t)\right] \\ &\quad = \cos\alpha_d Z_d T_{TM} f(x_2'\sin\alpha_d - c_d' t), \quad \forall x_2', t \end{aligned} \quad (10.6b)$$

From (10.6) one uniquely obtains

I) $dx_2'/dt = c_{\text{inc}}'/\sin\alpha = c_{sc}'/\sin\alpha_{sc} = c_d'/\sin\alpha_d$, which requires $\alpha_{sc} = \alpha$, $c_{sc}' = c_{\text{inc}}'$, $\omega_{sc}' = \omega_{\text{inc}}'$ and the extended Snell relation

$$\sin\alpha = n(1 \mp \beta\cos\alpha)\sin\alpha_d \quad (10.7)$$

II) the reflection and transmission coefficients

$$R_{TM} = \frac{Z\cos\alpha - Z_d\cos\alpha_d}{Z\cos\alpha + Z_d\cos\alpha_d}, \ T_{TM} = \frac{2Z\cos\alpha}{Z\cos\alpha + Z_d\cos\alpha_d} \quad (10.8)$$

The Brewster angle α_B for which one has zero reflection coefficient ($R_{TM} = 0$) is calculated from the equation $Z\cos\alpha = Z_d\cos\alpha_d$, which, upon substituting the extended Snell relation, shapes into the fourth order transcendental equation $Z_d^2(n^2 - \sin^2\alpha_B) = Z^2 n^2 \cos^2\alpha_B (1 \mp \beta\cos\alpha_B)^2$. A change of variables $\xi = \cos\alpha_B \in [0,1)$ provides a compact closed form representation

$$n^2 - 1 + \xi^2 - (Z/Z_d)^2 n^2 \xi^2 (1 \mp \beta\xi)^2 = 0 \ . \quad (10.9)$$

For $\forall n, \beta$ there is always one and only one root of the polynomial (10.9) that falls into the described range of ξ. For the special case $\mu_d = \mu$ one has $Z/Z_d = n$ and (10.9) simplifies as

$$n^4\beta^2\xi^4 \mp 2\beta n^4\xi^3 + (n^4 - 1)\xi^2 + 1 - n^2 = 0 \ . \quad (10.10)$$

A first order approximation in β requires $n^4\beta^2\xi^4 << 2\beta n^4\xi^3$, namely $\beta\xi << 2$, which can also be considered as roughly equivalent to $\beta < 0.2$. Under this condition (10.10) reduces to the cubic polynomial

$$\mp 2\beta n^4\xi^3 + (n^4 - 1)\xi^2 + 1 - n^2 = 0 \ . \quad (10.11)$$

(10.10) and (10.11) can always be solved uniquely for the described physical range of ξ by Cardano's analytical formulas for third and fourth order polynomials. The limiting case $\beta = 0$ yields the classical result $\xi = 1\big/\sqrt{n^2 + 1}$.

The total reflection mechanism $R_{TM} = 1$ is observed for $\alpha_d = \pi/2$ and the angle of total reflection α_{TR} is calculated from the relation $\sin\alpha_{TR} \pm n\beta\cos\alpha_{TR} = n$, which, by a change of variables $\varsigma = \sin\alpha_{TR} \in [0,1)$, can be shaped into the quadratic equation

$$\varsigma^2(1 + n^2\beta^2) - 2n\varsigma + n^2(1 - \beta^2) = 0 \ . \quad (10.12)$$

For a physical solution the discriminant of (10.12) requires to be positive: $\Delta = 4n^2\beta^2[1 - n^2(1 - \beta^2)] \ge 0$, or equivalently,

$$n \le 1\big/\sqrt{(1 - \beta^2)} \ . \quad (10.13)$$

Since $\beta \in [0,1)$, (10.13) can be satisfied for $n > 1$ when $\beta \ne 0$ as well. Under the condition (10.13) the angles of total reflection for $\vec{v} = +G\hat{x}_1$ and $\vec{v} = -G\hat{x}_1$ are found respectively, as

$$\alpha_{TR} = \sin^{-1}\left[\frac{n}{(1 + n^2\beta^2)}\left[1 - \beta\sqrt{1 - n^2(1 - \beta^2)}\right]\right] \quad (10.14a)$$

$$\alpha_{TR} = \sin^{-1}\left[\frac{n}{(1 + n^2\beta^2)}\left[1 + \beta\sqrt{1 - n^2(1 - \beta^2)}\right]\right]. \quad (10.14b)$$

Finally, the images of $\left(\vec{E}'_{sc},\vec{H}'_{sc}\right)$ and $\left(\vec{E}'_{D},\vec{H}'_{D}\right)$ in E-frame read

$$\begin{aligned}\vec{H}_{sc}(\vec{r};t) &= \hat{x}_3 R_{TM} f(\hat{n}_{sc}\cdot\vec{r}-c_{sc}t)\\ &= \hat{x}_3 R_{TM} g(k\hat{n}_{sc}\cdot\vec{r}-\omega_{sc}t)\end{aligned} \tag{10.15a}$$

$$\vec{E}_{sc}(\vec{r};t) = Z\vec{H}_{sc}(\vec{r};t)\times\hat{n}_{sc} \tag{10.15b}$$

$$\begin{aligned}\vec{H}_{d}(\vec{r};t) &= \hat{x}_3 T_{TM} f(\hat{n}_{d}\cdot\vec{r}-c_{tr}t)\\ &= \hat{x}_3 T_{TM} g(k_d\hat{n}_{d}\cdot\vec{r}-\omega_{tr}t)\end{aligned} \tag{10.16a}$$

$$\vec{E}_{d}(\vec{r};t) = Z_d\vec{H}_{d}(\vec{r};t)\times\hat{n}_{d} \tag{10.16b}$$

with

$$\hat{n}_{sc}\cdot\vec{r} = -x_1\cos\alpha + x_2\sin\alpha\ ,\ \hat{n}_d\cdot\vec{r} = x_1\cos\alpha_d + x_2\sin\alpha_d$$

$$c_{sc} = c(1\mp\beta\cos\alpha)\ ,\ c_{tr} = c_d(1\pm\beta n\cos\alpha_d)\ , \tag{10.17}$$

and the angular frequencies

$$\omega_{sc} = \omega_{inc}(1\mp 2\beta\cos\alpha)\ ,\ \omega_{tr} = \omega_{inc}(1\pm\beta n\cos\alpha_d)\ , \tag{10.18}$$

which reveal the famous Doppler effect.

10.2. Case II: Harmonic Motion

In this example we consider the special case of harmonic motion $\vec{\mathrm{v}}(t) = G(t)\hat{x}_1$, $G(t) = G\cos(\omega t)$, $G = const$ with coordinate transformations $x_1 = x'_1 + F(t)$, $F(t) = \left(G/\omega\right)\sin(\omega t)$, $x_2 = x'_2$, $x_3 = x'_3$; $\hat{x}_i = \hat{x}'_i$, $i = 1,2,3$ and under monochromatic TM plane wave incidence with

$$g = \cos(k\hat{n}_{inc}\cdot\vec{r}-\omega_{inc}t) = \mathrm{Re}\left\{e^{ik\hat{n}_{inc}\cdot\vec{r}}e^{-i\omega_{inc}t}\right\}\ . \tag{10.26}$$

In virtue of the well known Bessel property

$$e^{i\Omega\sin(\omega t)} = \sum_{-\infty}^{\infty} J_m(\Omega)e^{im\omega t}\ , \tag{10.27}$$

the image of (10.26) in L-frame is obtained as

$$\begin{aligned} g &= \sum_{-\infty}^{\infty}\mathrm{Re}\left\{J_m(\Omega)e^{ik\hat{n}'_{inc}\cdot\vec{r}'}e^{-i(\omega_{inc}-m\omega)t}\right\}\\ &= \sum_{-\infty}^{\infty}J_m(\Omega)\cos\left(k\hat{n}'_{inc}\cdot\vec{r}'-(\omega_{inc}-m\omega)t\right)\end{aligned} \tag{10.28}$$

with $\Omega = \left(G/\omega\right)k\cos\alpha$, which indicates an infinite sum of plane wave modes with amplitude $J_m(\Omega)$ and angular frequency $\omega'^{(m)}_{inc} = \omega_{inc} - m\omega$. Accordingly, the incident wave can be expressed as

$$\vec{H}'_{\mathrm{inc}}(\vec{r}';t) = \sum_{-\infty}^{\infty}\mathrm{Re}\left\{\vec{H}'^{(m)}_{\mathrm{inc}}(\vec{r}')e^{-i(\omega_{\mathrm{inc}}-m\omega)t}\right\} \tag{10.29a}$$

$$\vec{E}'_{\mathrm{inc}}(\vec{r}';t) = \sum_{-\infty}^{\infty}\mathrm{Re}\left\{\vec{E}'^{(m)}_{\mathrm{inc}}(\vec{r}')e^{-i(\omega_{\mathrm{inc}}-m\omega)t}\right\} \tag{10.29b}$$

$$\vec{H}'^{(m)}_{\mathrm{inc}}(\vec{r}') = \hat{x}'_3 J_m(\Omega)e^{ik\hat{n}'_{\mathrm{inc}}\cdot\vec{r}'}\ ,\ \vec{E}'^{(m)}_{\mathrm{inc}}(\vec{r}') = Z\vec{H}'^{(m)}_{\mathrm{inc}}(\vec{r}')\times\hat{n}'_{\mathrm{inc}} \tag{10.29c,d}$$

with

$$\hat{n}'_{\mathrm{inc}}\cdot\vec{r}' = x'_1\cos\alpha + x'_2\sin\alpha\ .$$

Based on the principle of superposition for sources and fields, the scattered field and the total field in region D can be given by

$$\vec{H}'_{sc}(\vec{r}';t) = \sum_{-\infty}^{\infty}\mathrm{Re}\left\{\vec{H}'^{(m)}_{sc}(\vec{r}')e^{-i\omega'^{(m)}_{sc}t}\right\} \tag{10.30a}$$

$$\vec{E}'_{sc}(\vec{r}';t) = \sum_{-\infty}^{\infty}\mathrm{Re}\left\{\vec{E}'^{(m)}_{sc}(\vec{r}')e^{-i\omega'^{(m)}_{sc}t}\right\} \tag{10.30b}$$

$$\vec{H}'^{(m)}_{sc}(\vec{r}') = \hat{x}'_3 J_m(\Omega)R^{(m)}_{TM}e^{ik\hat{n}'_{sc}\cdot\vec{r}'} \tag{10.30c}$$

$$\vec{E}'^{(m)}_{sc}(\vec{r}') = Z\vec{H}'^{(m)}_{sc}(\vec{r}')\times\hat{n}'_{sc} \tag{10.30d}$$

and

$$\vec{H}'_{d}(\vec{r}';t) = \sum_{-\infty}^{\infty}\mathrm{Re}\left\{\vec{H}'^{(m)}_{d}(\vec{r}')e^{-i\omega'^{(m)}_{d}t}\right\} \tag{10.31a}$$

$$\vec{E}'_{d}(\vec{r}';t) = \sum_{-\infty}^{\infty}\mathrm{Re}\left\{\vec{E}'^{(m)}_{d}(\vec{r}')e^{-i\omega'^{(m)}_{d}t}\right\} \tag{10.31b}$$

$$\vec{H}'^{(m)}_{d}(\vec{r}') = \hat{x}'_3 J_m(\Omega)T^{(m)}_{TM}e^{ik_d\hat{n}'^{(m)}_{d}\cdot\vec{r}'} \tag{10.31c}$$

$$\vec{E}'^{(m)}_{d}(\vec{r}') = Z_d\vec{H}'^{(m)}_{d}(\vec{r}')\times\hat{n}'^{(m)}_{d} \tag{10.31d}$$

with

$$\hat{n}'_{sc}\cdot\vec{r}' = -x'_1\cos\alpha_{sc} + x'_2\sin\alpha_{sc}\ ,\ \hat{n}'^{(m)}_{d}\cdot\vec{r}' = x'_1\cos\alpha^{(m)}_{d} + x'_2\sin\alpha^{(m)}_{d}\ .$$

The boundary and radiation conditions (10.5) uniquely yields

$$\omega'^{(m)}_{sc} = \omega'^{(m)}_{\mathrm{inc}} = \omega_{\mathrm{inc}} - m\omega\ ,\ \omega'^{(m)}_{d} = \omega_{\mathrm{inc}}\ ,\ \alpha_{sc} = \alpha\ ; \tag{10.32}$$

the Snell relation for each mode

$$\sin\alpha = n(1-m\omega/\omega_{inc})\sin\alpha^{(m)}_{d}\ ; \tag{10.33}$$

and the corresponding reflection and transmission coefficients

$$R^{(m)}_{TM} = \frac{Z\cos\alpha - Z_d\cos\alpha^{(m)}_{d}}{Z\cos\alpha + Z_d\cos\alpha^{(m)}_{d}}\ ,\ T^{(m)}_{TM} = \frac{2Z\cos\alpha}{Z\cos\alpha + Z_d\cos\alpha^{(m)}_{d}} \tag{10.34}$$

11. TE Plane Wave Scattering by a PEC Cylinder in Uniform Rotational Motion

Let us consider an incident monochromatic TE plane wave with electrical field with phasor

$$\vec{E}_{\mathrm{inc}}(\vec{r};t) = \hat{x}_3 e^{ikx_1}e^{-i\omega_{\mathrm{inc}}t} \tag{11.1}$$

impinging on an infinitely long PEC cylinder lying along x_3-axis, centered at origin and having radius a. The cylinder is assumed to rotate in counterclockwise direction with uniform angular velocity ω, which obeys the coordinate transformations rules outlined in Table 1 with $\phi(t) = \omega t$, $\vec{\mathrm{v}}(t) = \omega a\hat{\phi}(t)$. Setting $x_1 = \rho\cos\phi$, in virtue of (10.27), the incident field can be expressed as an infinite sum of modes

$$\vec{E}_{\text{inc}}(\vec{r};t) = \hat{x}_3 e^{ik\rho\cos\phi} e^{-i\omega_{\text{inc}}t} = \hat{x}_3 \sum_{-\infty}^{\infty} J_m(k\rho) e^{-im(\phi-\pi/2)} e^{-i\omega_{\text{inc}}t} = \sum_{-\infty}^{\infty} \vec{E}_{\text{inc}}^{(m)}(\vec{r};t) \quad (11.2)$$

Inserting the polar coordinate maps

$$\rho = \rho' \ , \ \phi = \omega t + \phi' \ , \quad (11.3)$$

the incident field has the L-frame representation

$$\begin{aligned}\vec{E}'_{\text{inc}}(\vec{r}';t) &= \hat{x}'_3 e^{ik\rho'\cos(\omega t+\phi')} e^{-i\omega_{\text{inc}}t} \\ &= \hat{x}'_3 \sum_{-\infty}^{\infty} J_m(k\rho') e^{-im(\phi'+\omega t-\pi/2)} e^{-i\omega_{\text{inc}}t} \\ &= \hat{x}'_3 \sum_{-\infty}^{\infty} J_m(k\rho') e^{-im(\phi'-\pi/2)} e^{-i\omega'^{(m)}_{\text{inc}}t} \\ &= \sum_{-\infty}^{\infty} \vec{E}'^{(m)}_{\text{inc}}(\vec{r}';t)\end{aligned} \quad (11.4)$$

where we define $\omega'^{(m)}_{inc} = \omega_{inc} + m\omega$. That the incident field satisfies the homogeneous frame indifferent wave equation

$$\left(lap' - \frac{1}{c^2} \frac{\lozenge'^2}{\lozenge' t^2} \right) \vec{E}'_{inc} = \vec{0} \quad (11.5)$$

can be seen upon the substitutions

$$lap' \vec{E}'_{inc} = lap' \left(\hat{x}'_3 e^{ik(x'_1\cos(-\omega t) + x'_2 \sin(-\omega t))} e^{-i\omega_{inc}t} \right) = -k^2 \vec{E}'_{inc}$$

$$\vec{v}'(\vec{r}';t) = -\omega\rho'\hat{\phi}'(t) \ , \ \hat{\phi}'(t) = \hat{x}'_1 \cos(-\omega t) + \hat{x}'_2 \sin(-\omega t)$$

$$\vec{v}' \cdot grad' = -\omega \frac{\partial}{\partial \phi'} \ , \ \frac{\lozenge'}{\lozenge' t} = \frac{\partial}{\partial t} - \omega \frac{\partial}{\partial \phi'}$$

$$\frac{\lozenge'}{\lozenge' t} \vec{E}'_{inc} = -i\omega_{inc} \vec{E}'_{inc} \ , \ \frac{1}{c^2} \frac{\lozenge'^2}{\lozenge' t^2} \vec{E}'_{inc} = -\frac{\omega_{inc}^2}{c^2} \vec{E}'_{inc} = -k^2 \vec{E}'_{inc}$$

Based on the principle of superposition for sources and fields, the scattered field $\vec{E}'^{(m)}_{sc}(\vec{r}';t)$, corresponding to the m-th mode of incidence, is to be calculated from the boundary value problem

$$\begin{cases} \left(lap' - \frac{1}{c^2} \left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \phi'} \right)^2 \right) \vec{E}'^{(m)}_{sc}(\vec{r}';t) = \vec{0}, \rho' > a \\ \text{Boundary Cond}: \vec{E}'^{(m)}_{sc} + \vec{E}'^{(m)}_{inc} = \vec{0}, \ \ \rho' = a, \ \forall \phi' \\ \text{Periodicity Cond}: \vec{E}'^{(m)}_{sc}(\rho' = a, \phi') = \vec{E}'^{(m)}_{sc}(\rho' = a, \phi' + 2\pi), \ \forall \phi' \\ \text{Radiation Condition as } \ \rho' \to \infty \end{cases} \quad (11.6)$$

In virtue of the analytical structure of $\vec{E}'^{(m)}_{inc}$, we may apply the method of separation of variables as

$$\vec{E}'^{(m)}_{sc} = R^{(m)}(\rho') \Phi^{(m)}(\phi') e^{-i\omega'^{(m)}_{\text{inc}}t} \quad (11.7)$$

The angular component satisfies the reduced boundary value problem

$$\ddot{\Phi}^{(m)}(\phi') + \nu^2 \Phi^{(m)}(\phi') = 0 \quad , \quad \Phi^{(m)}(\phi') = \Phi^{(m)}(\phi' + 2\pi) \quad (11.8)$$

which enforces the separation constant ν to take positive integer values $\nu = 1, 2, ...$ while we choose $\Phi_\nu^{(m)}(\phi') = e^{-i\nu(\phi'-\pi/2)}$.

The radial component satisfies the Bessel equation

$$\left(\frac{d^2}{d\rho'^2} + \frac{1}{\rho'} \frac{d}{d\rho'} + \left(\frac{\omega'^{(m-\nu)}_{\text{inc}}}{c} \right)^2 - \frac{\nu^2}{\rho'^2} \right) R_\nu^{(m)} = 0 \ , \quad (11.9)$$

which uniquely yields ν-th order Hankel functions of the first kind $H_\nu^{(1)}\left(\frac{\omega'^{(m-\nu)}_{\text{inc}}}{c} \rho' \right)$ under the radiation condition. Accordingly, the sought for scattered field can be written as

$$\vec{E}'^{(m)}_{sc}(\vec{r}';t) = \hat{x}'_3 \sum_{\nu=1}^{\infty} a_\nu^{(m)} H_\nu^{(1)} \left(\frac{\omega'^{(m-\nu)}_{\text{inc}}}{c} \rho' \right) e^{-i\nu(\phi'-\pi/2)} e^{-i\omega'^{(m)}_{\text{inc}}t} \ , \quad (11.10)$$

where the unknown coefficients $a_\nu^{(m)}$ are to be solved from the reduced boundary relation

$$\sum_{\nu=1}^{\infty} a_\nu^{(m)} H_\nu^{(1)} \left(\frac{\omega'^{(m-\nu)}_{\text{inc}}}{c} a \right) e^{-i\nu(\phi'-\pi/2)} = -J_m(ka) e^{-im(\phi'-\pi/2)} \ , \quad \forall \phi' \quad (11.11)$$

Based on the orthogonality property of the trigonometric functions as

$$\int_0^{2\pi} e^{+ir(\phi'-\pi/2)} e^{-i\nu(\phi'-\pi/2)} d\phi' = \begin{cases} 0, \nu \neq r \\ 2\pi, \nu = r \end{cases},$$

one can multiply both sides of (11.11) by $e^{+ir(\phi'-\pi/2)}$ and integrate w.r.t. ϕ' from 0 to 2π to get

$$2\pi a_r^{(m)} H_r^{(1)} \left(\frac{\omega'^{(m-r)}_{inc}}{c} a \right) = -J_m(ka) \int_0^{2\pi} e^{+ir(\phi'-\pi/2)} e^{-im(\phi'-\pi/2)} d\phi' = \begin{cases} 0, m \neq r \\ -2\pi J_m(ka), m = r \end{cases}$$

which reads

$$a_\nu^{(m)} = \begin{cases} 0, \nu \neq m \\ -J_m(ka) / H_m^{(1)}(ka), \nu = m \end{cases} \quad (11.12)$$

and eventually

$$\vec{E}'^{(m)}_{sc}(\vec{r}';t) = -\hat{x}'_3 \frac{J_m(ka)}{H_m^{(1)}(ka)} H_m^{(1)}(k\rho') e^{-im(\phi'-\pi/2)} e^{-i\omega'^{(m)}_{\text{inc}}t} . \quad (11.13)$$

$$\vec{E}^{(m)}_{sc}(\vec{r};t) = -\hat{x}_3 \frac{J_m(ka)}{H_m^{(1)}(ka)} H_m^{(1)}(k\rho) e^{-im(\phi-\pi/2)} e^{-i\omega_{\text{inc}}t} \quad (11.14)$$

$$\vec{E}_{sc}(\vec{r};t) = -\hat{x}_3 \left[\sum_{-\infty}^{\infty} \frac{J_m(ka)}{H_m^{(1)}(ka)} H_m^{(1)}(k\rho) e^{-im(\phi-\pi/2)} \right] e^{-i\omega_{\text{inc}}t} . \quad (11.15)$$

It is observed that the total scattered field is monochromatic, having the same frequency as the incident field and its expression is independent of the frequency of rotation, coinciding with the result for the stationary case ($\omega = 0$).

12. Conclusion

The present investigation is planned to pursue by involving more boundary values of practical interest in a systematic manner. It should also be interesting to discuss the alternative solutions delivered by Special Relativity Theory of Einstein (SRT), both conceptually and numerically, for the same sets of boundary value problems, whenever SRT applies in its own description.

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