

# Zero velocity must be relative

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## Abstract

The central concept of the theory of relativity is **relative** velocity. The velocity of a material body is not an intrinsic property of this body; it depends on the free choice of reference system. Relative velocity is thus reference-dependent, it is not an absolute concept. We stress that even zero-velocity must be relative. Every reference system possess its own zero-velocity relative to exactly that one system. The theory of relativity in terms of relative velocities, with many zero-velocities, is formulated within an associative groupoid structure, which is *neither* a group, *nor* a non-associative loop. Moreover, we discuss conceptual dichotomy: two different rival concepts of reference system: the Minkowski space-time observer-monad (*i.e.* a time-like vector field), versus the Einstein space-time coordinate tetrad.

**Keywords:** reference frame, monad versus tetrad, observer, relativity groupoid as a category

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# 1 Velocity is relative

Galileo observed in 1632 that every velocity is relative and our everyday experience tells us the same thing. One can not discover one's own movement without looking outside for another reference system.

The theory of relativity (of Galileo 1632, and, also of Einstein 1905 and Minkowski 1908) is about the concept of relative velocity. We avoid the historical name *special relativity* introduced by Einstein in 1916 in order to distinguish *special* (not gravity) from *general relativity* ( $\equiv$  gravity). The wording 'general relativity' is synonymous with the *theory of gravity*. Einstein introduced this now obsolete terminology being motivated by the role of coordinate systems. The word 'special' for particular coordinates, and 'general' for arbitrary coordinates. We share the opinion that coordinates are irrelevant for physics. However there is other opinion that coordinates are not always a mathematical convention, and is argued that the Global Positioning System GPS is example where coordinate system is *basic physics*, see [Coll 2000, Rovelli 2003 §2.4.6]. In the present paper relativity means the historical term 'special relativity', where we drop 'special' because the theory of relativity, in our understanding, is *coordinate-free*.

The velocity of one massive body is always relative to other body, or, we could say that the velocity of a body is always relative to a free *choice* of reference system. In fact relativity theory is a theory of reference systems. The historical name 'special relativity' must be interpreted precisely as 'velocity is relative', or, 'velocity depends on a free *choice* of reference system'. To be *relative* means that one must keep information about what specific target-body has the given velocity relative to what source-body. A velocity of a bus relative to the street is denoted by  $\mathbf{v}(\text{street} \rightarrow \text{bus})$ . We postulate that every velocity must be a function of a two-body material system.

**1.1 Notation** (Target- and source- body). A target-body is an **observed** body seeming to possess this velocity, and a source-body is an **observer** laboratory, a reference body, measuring this relative velocity. The relative velocity we will denote as follows

$$\mathbf{v}_{ST} \equiv \mathbf{v}(\text{source} \rightarrow \text{target}), \quad \text{OR} \quad \mathbf{v}_{\text{source} \rightarrow \text{target}} \quad \text{OR} \quad \mathbf{v}_{\text{target} \leftarrow \text{source}} \quad (1)$$

If we do not define or make precise the meaning of material body, then the concept of velocity is meaningless. The very concept of a material body

is most crucial for understanding the concept of velocity. Georg Hegel (1770–1831) wrote: *no motion without matter*.

Most definitions of velocity are obscured by the coordinate system. These coordinate systems contain implicit, hidden or incomplete, obscure information about material bodies.

Light is massless and therefore can not be considered to be a reference system this includes cosmic background radiation. **The velocity of light must *not* be considered a primary concept of relativity theory.** The similar opinion is shared by [Paiva and Ribeiro 2005] that *special relativity does not depend on electromagnetism*. Here is subtle difference, the relativity does not postulate on light velocity, however needs invertible metric tensor on spacetime and this metric tensor seems to be involved in Maxwell electromagnetism.

**1.2 Axiom** (The first axiom of relativity theory). We consider the following three terms as synonymous: a material body, an observer, a reference system. What is a material body in a spacetime? A tetrad, frame field, vierbein also known as the Lorentz basis. Or: a monad field, a reference fluid as a time-like vector field on spacetime.

The definition of relative velocity we are going to introduce here in §8, does not depend on the existence or absence of a privileged reference system or æther. The extra assumption of a constant relative velocity is not important, the inertial-system is also of secondary relevance.

## 2 Zero velocity is relative: groupoid versus group

Can we add every *relative* velocity with every other *relative* velocity? We declare that the velocity of the Sun relative to the Earth,  $\mathbf{v}(\text{Earth} \rightarrow \text{Sun})$ , and the velocity of Mars relative to Jupiter,  $\mathbf{v}(\text{Jupiter} \rightarrow \text{Mars})$ , are not composable, being a meaningless addition. We conclude that the composition of **relative** velocities is not always possible.

We wish each zero velocity also be a relative velocity. Therefore there must be as many relative zero-velocities as there are reference systems. We do not wish to identify the zero-velocity of the Sun relative to the Sun,  $\mathbf{0}(\text{Sun} \rightarrow \text{Sun})$ , with the zero-velocity of a bus relative to that bus,  $\mathbf{0}(\text{bus} \rightarrow \text{bus})$ . We do not accept the identification of zero-velocities for different observers.

Thus the set of all relative velocities, not all of which are composable, can not be a group. Rather it is a groupoid, a *relativity groupoid*. The mathematical concept of a groupoid (generalizing a group) was introduced by Brandt in 1926 (the same year quantum mechanics was born) therefore groupoid is commonly known as Brandt groupoid [Brandt 1926, Ehresmann 1957, Mordijk and Pronk 1997]. The Brandt groupoid is a particular category where every morphism is an isomorphism. A groupoid with exactly one object (one neutral ‘unit’ morphism) is said to be a group, and in this particular case all isomorphisms are composable.

**2.1 Notation** (Every coordinate is a scalar field). In what follows a unital associative and commutative  $\mathbb{R}$ -algebra of scalar fields on space-time of events is denoted by  $\mathcal{F}$ ,

$$t, x, y, z, \frac{\partial t}{\partial t} = 1, \frac{\partial x}{\partial t} = 0 \in \mathcal{F}, \quad i.e., \text{ if } e \text{ is an event then } t(e) \in \mathbb{R}. \quad (2)$$

Every coordinate, like,  $t, x, y, z$ , is a particular scalar field, because it is assumed that,

$$\frac{\partial t}{\partial t} = 1 \in \mathcal{F}, \quad \frac{\partial x}{\partial t} = 0 \in \mathcal{F}, \quad \frac{\partial y}{\partial t} = 0 \in \mathcal{F}, \quad \frac{\partial z}{\partial t} = 0 \in \mathcal{F}, \quad \text{etc.} \quad (3)$$

A vector field is synonymous to a *derivation* of algebra of scalar fields. A set of all vector fields, denoted by  $\text{der } \mathcal{F}$ , is a  $\mathcal{F}$ -Lie-module, a ‘module’ generalize the concept of a vector space<sup>1</sup>.

Returning to the concept of a groupoid versus group let us see how the Lorentz group is a particular groupoid. The Lorentz relativity *group* is an isometry group of space-time. The domain of isometry are vector fields. Every isometry must be an endomorphism of  $\text{der } \mathcal{F}$ , isometry ‘permutes’ vector fields on space-time,

$$\text{isometry} \in \text{End } \text{der } \mathcal{F}. \quad (4)$$

Therefore the entire *module* of all vector fields on space-time,  $\text{der } \mathcal{F}$ , is precisely *one* object for the Lorentz *group*.

In contrast, the relativity groupoid possesses as many objects as there are massive bodies in mutual motions.

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<sup>1</sup>The Lie-module structure is so important that it is known under multitude of other names, Lie-Rinehart algebra, Lie-Cartan pair, pseudo-algèbre de Lie, Lie algebroid,  $\mathcal{F}$ -Lie algebra  $\text{der } \mathcal{F}$ , etc.

### 3 Reference system: monad versus tetrad

It is not widely recognized among relativity researchers that there is no unique mathematical conception of the meaning of a material body as a reference system. Here is the crucial dichotomy:

Albert Einstein introduced one concept of a reference system in 1905, identifying a physical reference system with the mathematical coordinate system consisting of scalar fields on *space-time of events*, the coordinate system ‘attached’ to a observer-body. The Einstein concept gives rise a posterior to the name **tetrad** to refer to a basis in four-dimensional vector space, see (12)-(B) below.

An alternative definition of deformable material body comes from fluid mechanics and from fluid dynamics in terms of partial derivatives. It is also known as the Euler derivative, alias substantial or material, or barycentric, or hydrodynamic derivative. In the fluid context a material body is not an Einsteinian set of scalar coordinate fields, (2)-(3), but rather a **vector** field on space-time, an event-dependent (‘space’ and ‘time’-dependent) vector field  $\in \text{der } \mathcal{F}$ .

Within the relativity of simultaneity it was Hermann Minkowski who defined in 1908 a reference system as the time-like vector field on space-time [Minkowski 1908 §4, formula (19)]. This Minkowski reference fluid was posterior baptized by Eckart in 1940 as a *perfect fluid*, and by Zel’manov in 1976 as a *monad* field. See [Eckart 1940, Zel’manov 1976, Mitskievich 2006, Rodrigues and Capelas 2007], among many other.

Each vector field is a synonym of a derivation of an algebra, and therefore each monad reference system must be seen as a derivation of space-time algebra,  $\in \text{der } \mathcal{F}$ . The coordinate expression (in adopted coordinates) for the reference fluid is given in a form well known from thermodynamics textbooks in analogy to isobaric processes, whereas within the space-time of events, a reference fluid is an iso- $(x, y, z)$  process (3),

$$Vt \equiv 1 \quad \text{and} \quad Vx \equiv 0 \quad \iff \quad V \simeq \left( \frac{\partial}{\partial t} \right)_{x,y,z} \in \text{der } \mathcal{F}. \quad (5)$$

Two concepts of material body, the Einsteinian tetrad and the reference spacetime-fluid monad, are different. The tetrad, also called the vierbein, the frame field, the Lorentz tetrad, do not need to include time-like vector, can include two time-like vectors (Gödel’s metric), can be built from light-like vectors + some space-likes (thus where is a mass?). If tetrad consists

of exactly one time-like vector + three other (orthogonal) space-like vectors, then *another* tetrad with exactly the same time-like vector + triple of another space-like vectors, we identify with exactly the same massive body (rotated). Within monad-field a rotation of a body (non-inertial) is encoded inside of the covariant derivative of the monad. Only important is time-like vector-field alone if it is a member of a tetrad. Tetrad with one time-like vector-field is, in our opinion, equivalent to the monad-field alone, and one can forget about basis-frame. Compare with discussions in [Arminjon and Reifler 2010, Landau and Lifshitz 2006, Llosa and Soler 2004]. Adherents of tetrads probably see virtue in the treatment all vectors on the same footing, as they must be for an isometry. So light-like vector is as ‘good’ vector as other vectors, maybe even more important as light is considered to be indispensable for the measurements of distances. Whereas the monads do not accept light-like vector-fields as the reference systems, and this allows permutation of reference systems in terms of *non-isometric* groupoid for which light-like light is outside of the domain.

Different concepts of material body implies that the concept of the relative velocity between two bodies is also different, depending on what concept of reference system (= material body) is chosen as the *first axiom* of relativity theory, axiom 1.2. Different mathematical conceptual models of a material body lead to different mathematical concepts of relative velocity, and give rise to distinct relativity theories.

In this respect there is not such a big difference between absolute Galilean simultaneity and relative simultaneity (= metric-dependent proper-time). We illustrate the above dichotomy of massive body using the example of Galilean absolute simultaneity, with *relative* spaces, cf. Figures 1 and 2.

In 1994 Tamás Matolcsi published a monograph *Space-time Without Reference Frames*, (the first version in 1984), where ‘without reference frame’ must be read ‘without Einsteinian coordinate system’, but **with** the concept of a reference system as the Euler or Minkowski monad. The first part, Part I, of Matolcsi’s monograph, 30% of the entire book, is devoted to Galilean absolute simultaneity called ‘Non-relativistic Space-Time Model’ with absolute simultaneity. ‘Non-relativistic’ must be interpreted as ‘relative spaces’.

## 4 Galilean relativity of space

Galileo Galilei observed in 1632 that to be in the same place is *relative*, subjective, not objective. Place is observer-dependent. Galileo conceptually introduced a four-dimensional *physical* space-time of absolute events, after the experimentally-based conclusion that it is impossible to detect the motion of a boat without a choice of outside reference system. If the concept of place in a three-dimensional space needs an artificial choice of some irrelevant reference system, then three-dimensional space is an illusion. Different reference systems possess different three-dimensional spaces, and the only objective physical-arena is four-dimensional space-time (Galilean or special-relativistic of Minkowski). Three-dimensional reference-dependent space is a *mathematical convention* that depends on a subjective choice of reference system. According to Galileo: there are as many three-dimensional spaces as there are reference systems, *i.e.* there does not exist a ‘unique physical space’, this ‘unique’ is ghost-space. Therefore space-time must not be seen as it was by the Aristotelan Greeks: an Earth-space moving in time. Aristotle has only one unique observer: the Earth. Galilean space-time allows infinite number of observers.

Galilean space-time is a fiber=simultaneity-bundle over one-dimensional time, without any preferred space [Trautman 1970]. Trautman suggests incorrectly that each fiber over a time-moment is ‘isomorphic to Euclidean 3-space  $\mathbb{R}^3$ ’, that one can interpret a fiber over a time as (isomorphic to) a physical space. This is not the case! Each fiber is a set of simultaneous **events**, and not a set of places in a ‘physical’ space! There is no space concept within Galilean physical space-time, because the concept of the space needs an artificial *choice* of the reference system. Galilean space-time is *not* the cartesian product of time with some ‘ghost-space’, because there does not exist privileged space among many spaces. There does not exist just a single space, there are *many* spaces.

If some reference system is chosen, Earth or Sun?, then the corresponding space of *this* massive body is *not* a fiber in space-time, but it is rather a quotient-space = space-time/material-body,

$$\text{Time} \equiv \frac{\text{Space-time}}{\text{Convention of simultaneity}}, \quad \text{Space} \equiv \frac{\text{Space-time}}{\text{material body}} \quad (6)$$

$$\text{Proper-Time} \equiv \frac{\text{Space-time}}{\text{Metric simultaneity of material body}}. \quad (7)$$

Galileo Galilei in our interpretation: *physical* reality is a four-dimensional space-time of **events**. Time can never stop, and a choice of three-dimensional space is no more than a *mathematical* convenience. The name *space-time*, introduced by Hermann Minkowski in 1908, is misleading, suggesting incorrectly that this concept is derived from two primitive concepts of ‘space’ and ‘time’. It is just the opposite, the most primitive concept is the Galilean space-time of events, and space is a *derived* concept that needs an artificial *choice* of massive body, e.g. Earth or Sun, as a reference system (6). But such a choice is irrelevant for all physical phenomena, it is no more than maybe a convenient mathematics for a computer program.

This is Galilean relativity: three-dimensional space does not exist as a physical reality, it is a mathematical convention. The Galilean four-dimensional space-time does not possess an invertible metric tensor. The Hermann Minkowski version of Einstein’s special relativity added an invertible metric tensor, the Minkowski metric, to Galilean space-time. For Galilean electrodynamics à la Minkowski in low velocities we refer to [Rousseaux 2008].

Galilean relativity postulates an absolute *simultaneity relation*, denoted by  $\tau$  on Figure 1. Composed with a clock-function it gives a coordinate on spacetime of events,

$$t = \text{clock} \circ \tau. \quad (8)$$

There is no need for another clock  $t' = t$ . Absolute simultaneity is perfectly compatible with Einstein and Minkowski special relativity where it can be identified as just one among many different conventions of synchronization, such as for example the radio-synchronization which gives simultaneity that is metric-free.

**4.1 Definition (Place).** Each reference system is completely defined in terms of an equivalence relation on events being in the *same* place.

Thus every observer-monad field,  $V \in \text{der } \mathcal{F}$ , gives rise to a surjective projection  $\pi_V$  from four-dimensional space-time of events, onto three-dimensional relative quotient  $V$ -space of places. Two space-time events,  $e_1$  and  $e_2$ , are in the same **place** for an  $\pi$ -observer if and only if,  $\pi(e_1) = \pi(e_2)$ .

**4.2 Example.** We must see how two reference systems, say a bus  $B$  and a street  $S$ , in a mutual motion, are distinguished within the space-time of **events**. Let us denote a street by  $\pi_S$ -system, and a bus by  $\pi_B$ -system. Let us illustrate the relativity of space in terms of the following list of three events:

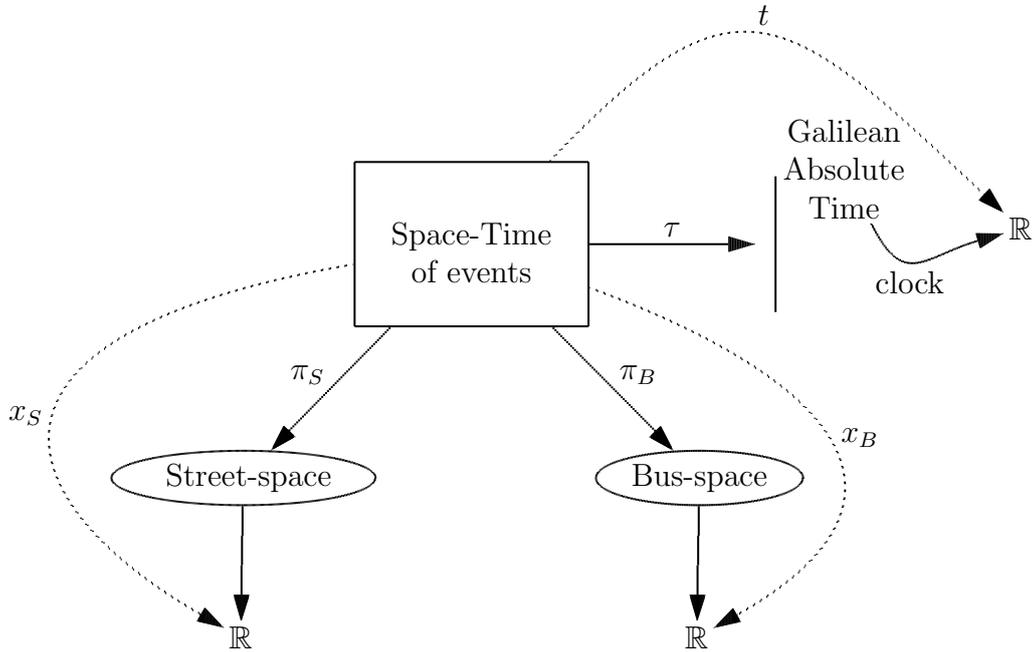


Figure 1: Two-body system, {Street, Bus}. To be in the same place is relative. Quotient  $B$ -space is different from quotient  $S$ -space.

- $e_1 =$  bus start from the bus stop ‘Metro’
- $e_2 =$  bus almost arrive to the next bus stop ‘Center’
- $e_3 =$  late passenger arrived to the bus stop ‘Metro’

From the point of view of a driver of the bus, this is  $\pi_B$ -system, driver is on the same  $B$ -place inside of the bus, bus is at  $\pi_B$ -‘rest’:

$$\pi_B(e_1) = \pi_B(e_2), \quad \text{but} \quad \pi_B(e_3) \neq \pi_B(e_1). \tag{9}$$

From the point of view of the crowd standing on the street, the street is  $\pi_S$ -system:

$$\pi_S(e_1) = \pi_S(e_3), \quad \text{but} \quad \pi_S(e_2) \neq \pi_S(e_1). \tag{10}$$

**4.3 Example.** Another example is a space of Sun and a space of Earth (Copernicus versus Ptolemy). The events are

- $e_1 =$  Greg born (in Long Beach in July)
- $e_2 =$  Bill born (in Long Beach in January)
- $e_3 =$  Jamie born (in Washington in July)

Were any of them, Greg, Bill, Jamie, born ‘in the same place’?

Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows and only a kind of union of two will preserve an independent reality.

Hermann Minkowski 1908

The Minkowski ‘union of two’ could suggest incorrectly uniqueness of space, and spacetime as a kind of Cartesian product of space by time. We must not insist on the necessity of the relativity of time, because the relative time is metric-dependent proper-time only, and nature allows other conventions of metric-free simultaneities, such as radio-simultaneity, etc. The primary concern of relativity theory is the obligatory relativity of *space*. Relativity of space is metric-free, and is simultaneity-free. This is observed also by Arminjon and Reifler [2010 §4 Discussion ii) on page 10].

For related explications we refer to [Trautman 1970; Matolcsi 1994 Part I §3].

Carlo Rovelli motivated by quantum theory concluded

The world in which we happen to live can be understood without using the notion of time.

Carlo Rovelli 2003, page 21

## 5 Material body as a coordinate system

In a *coordinate* reference system, a coordinate body  $S$ , is identified with a pair of coordinate scalar functions on the space-time of events,  $S \simeq \{\tau, \pi_S\} \simeq \{t, x_S\}$ , and another body is  $B \simeq \{\tau, \pi_B\} \simeq \{t, x_B\}$ , etc., where

$$t = \text{clock} \circ \text{simultaneity}, \quad x_S = f_S \circ \pi_S, \quad x_B = f_B \circ \pi_B, \quad (11)$$

$$\{\text{Space-Time of events}\} \xrightarrow{t, x_S, x_B} \mathbb{R}. \quad (12)$$

The above coordinate body in an Einsteinian reference system, is called a **tetrad**, and is given by a dual pair, the co-frame and the frame,

$$\{t, x\} \implies \{dt, dx\} \iff \left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right\} \quad (13)$$

Within coordinate bodies each reference system is defined in terms of a scalar function,  $x_S$ -system is a street, and  $x_B$ -system is a bus. Here  $x_S$  and  $x_B$  are functions on space-time of events, and two events,  $e_1$  and  $e_2$  are in the same **place** for an  $x$ -observer if and only if,  $x(e_1) = x(e_2) \in \mathbb{R}$ .

In this section our aim is to define relative velocity in terms of a ('material, massive') body identified with the coordinate system on a two-dimensional space-time of events. We need space-time, because it is not possible to stop time. Events float in spacetime.

Before defining the velocity of the bus relative to the street, we must suppose that we know a priori perfectly our coordinate bodies,  $x_S$  and  $x_B$ . We need the concept of a **relative** velocity among a pair of coordinate bodies. There is no concept of relative velocity without a concept of material body!

Instead, consider the 'well known' concept that abstract velocity is nothing but a 'group-parameter  $v$ ' in some adored group,  $x \mapsto x + vt$ . A one common dogmatic practice is to define velocity in terms of a boost transformation. This virus-dogma insists that the concept of velocity can be defined somehow independently from the very concept of reference system. All group elements can be composed and all such abstract group parameters can be added. The Galilean boost,  $\{\text{Group}; \text{Body}\} \equiv \{v; t, x_A\} \mapsto x_A + vt$ , presupposes that the abstract concept of the velocity-parameter  $v$  is defined a priori. The name boost was introduced by Eugene Wigner in 1939, and this name suggests a dynamical process of acceleration. But there is no dynamical Newton second law within the concept of boost, with its silently constant Einstenian velocity.

The velocity of the bus relative to the bus is zero, and velocity of the

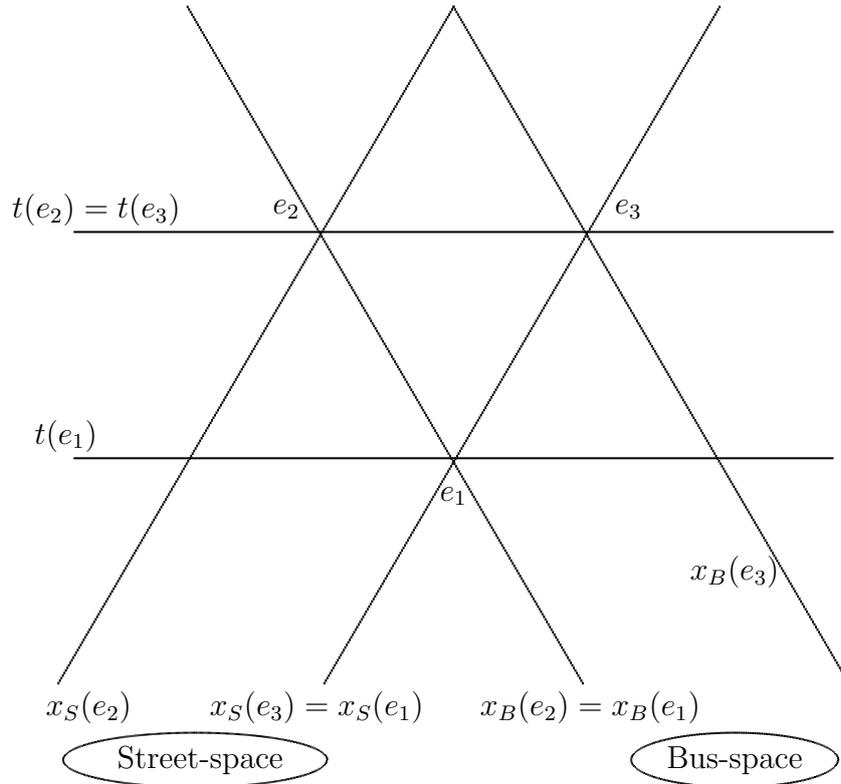


Figure 2: Two-body coordinate systems within absolute simultaneity,  $t(e_2) > t(e_1) \in \mathbb{R}$ . To be in the same place is relative.  $B$ -places are different from  $S$ -places. Entire space-time line  $(x_B)^{-1}(x_B e_3)$  denotes a **one** single  $B$ -place, it is a integral curve of the  $B$ -observer, see [Matolcsi 1994 Part I §3.2.1], and formula (23) in Section 6 here.

street relative to the street is also zero,

$$\begin{aligned}
 v_{BB}(e_2, e_1) &= \frac{\text{distance}}{\text{time}} \\
 &= \frac{x_B(e_2) - x_B(e_1)}{t(e_2) - t(e_1)} = 0, \\
 v_{SS}(e_3, e_1) &= \frac{x_S(e_3) - x_S(e_1)}{t(e_3) - t(e_1)} = 0,
 \end{aligned} \tag{14}$$

$$\begin{aligned}
 v_{S \rightarrow B}(e_3, e_2, e_1) &= \frac{x_S(e_2) - x_S(e_1)}{t(e_3) - t(e_1)}, \\
 v_{B \rightarrow S}(e_2, e_3, e_1) &= \frac{x_B(e_3) - x_B(e_1)}{t(e_2) - t(e_1)}, \\
 v_{S \rightarrow B}(e_3, e_2, e_1) + v_{B \rightarrow S}(e_2, e_3, e_1) &= 0 \in \mathbb{R}.
 \end{aligned} \tag{15}$$

As we see in (14)-(15), relative velocity uses only one coordinate, one observer only,  $x_S$  or  $x_B$ , but all three absolute events.

However if we suppose additionally the following initial conditions,

$$t(e_1) = 0, \quad \text{and} \quad x_S(e_1) = x_B(e_1), \quad (16)$$

then one can re-express relative velocity in terms of only one event but two coordinate-observers. This leads to velocity as a **scalar function on space-time** (and not on the relative-space of individual places),

$$\begin{aligned} v_{S \rightarrow B}(e_3, e_2, e_1) &= \frac{x_S(e_2) - x_B(e_2)}{t(e_2)} = \frac{x_S - x_B}{t} e_2 = v_{S \rightarrow B}(e_2), \\ v_{B \rightarrow S}(e_2, e_3, e_1) &= \frac{x_B(e_3) - x_S(e_3)}{t(e_3)} = \frac{x_B - x_S}{t} e_3 = v_{B \rightarrow S}(e_3). \end{aligned} \quad (17)$$

What is the physical meaning of the Galilean group parameter? If  $\{t, x_S\}$  describes a coordinate body, then  $x_S + vt$  must describe another massive body  $x_B$  that moves at a given speed  $v$  relative to the observer-street  $x_S$ . The link-problem clarifies the physical meaning of this innocent group-parameter  $v$  as relative velocity. The group-boost gives rise to the **definition** of relative velocity between two reference systems, as a solution of this link-problem,

$$\text{given } x_S \text{ and } x_B, \quad x_B = x_S + vt \quad \implies \quad v(x_B, x_S) = \frac{x_B - x_S}{t}. \quad (18)$$

**5.1 Bad intuition** (Galilean ‘relative velocity’). The Galilean relative velocity of a coordinate body  $x_B$  as observed (measured, as seen) by a coordinate body  $x_S$ , can be defined as follows

$$v_{SB} \equiv \frac{x_S - x_B}{t}, \quad v_{SB} + v_{BC} = v_{SC}. \quad (19)$$

In (19) a **scalar** field  $v_{SB}$  represents the ‘relative velocity’ of  $B$  as seen by  $S$ . This is á la Einstein, the relative velocity defined as a scalar parameter relating coordinate bodies.

**5.2 Comment** (Addition or composition?). The binary operation ‘+’ symbolizing relative-velocity addition in (19) is misleading because ‘+’ suggests incorrectly that this is commutative addition. This is not the case. The composition of relative velocities needs an *ordered* pair of relative velocities! If  $v_{SB}$  is the velocity of a bus  $B$  relative to the street  $S$ , and  $v_{BC}$  is the

velocity of a car  $C$  relative to this bus, one can compose,  $v_{CB} \circ v_{BA} = v_{CA}$ , in analogy to non-commutative composition of maps,  $g \circ f \neq f \circ g$ . In this ordering convention  $v_{BS}$  is composed with  $v_{CB}$ , we read composition from right to left as in Arabic language. Therefore the symbol of composition ‘ $\circ$ ’ must be more adequate because order of summands is most crucial:

$$\begin{aligned} \mathbf{v}_{C \leftarrow B} \circ \mathbf{v}_{B \leftarrow S} &= \mathbf{v}_{C \leftarrow S} \\ \mathbf{v}_{S \rightarrow B} \circ \mathbf{v}_{B \rightarrow C} &= \mathbf{v}_{S \rightarrow C} \end{aligned} \quad (20)$$

**5.3 Comment** (How to distinguish zero-velocities?). We wish to have a concept of massive body  $A \simeq v_{AA}$ , that will automatically distinguish the zero-velocities,  $v_{AA} \neq v_{BB}$  for  $v_{AB} \neq 0$ . However the bad-intuition in (19) implies  $v_{AA} = v_{BB} = \dots = 0$ . What is wrong in (17)-(19) is that the relative velocity is considered to be a scalar function on **space-time**, whereas we must try to see these relative velocities on individual relative quotient-spaces.

Every Eulerian observer-fluid  $A \in \text{der } \mathcal{F}$  (23) gives rise to a surjective projection  $\pi_A$  from four-dimensional space-time of events, onto three-dimensional relative  $A$ -space,  $A \circ \pi_A^* \equiv 0$ .

With this in mind the following substitutions, must be used in all expressions above,

$$\begin{aligned} x_A &= f \circ \pi_A, & v_{AB} &\rightsquigarrow v_{AB} \circ \pi_A, \\ x_B &= g \circ \pi_B, & v_{BA} &\rightsquigarrow v_{BA} \circ \pi_B. \end{aligned} \quad (21)$$

In this case the interpretation of the group parameter  $v$  within the relative quotient-space needs to be re-thought, *i.e.* how can we get rid off the different projections,  $\pi_A \neq \pi_B$ .

**5.4 Comment** (Group or groupoid?). Can we consider the set of all *relative* velocities to be a group? To be **relative** means that velocity must keep information of what concrete target-body the velocity is relative to what source-body, therefore there must be as many zero-velocities as reference systems  $A \leftrightarrow 0_A$ ,  $v_{AA} \equiv 0_A \neq 0_B \equiv v_{BB}$ . Each individual  $A$ -quotient-space is expected to possess its own zero velocity, exactly  $0_A$ .

Consider four reference systems (for simplicity in two-dimensional space-time) given by space-time coordinates:  $x_A, x_B, x_Y, x_Z$ , appropriately related to the notation in Figure 3. The false relative velocities tangent to iso-simultaneity instantaneous three-dimensional subevents, not yet projected onto relative quotient-spaces of places, can all be composed. But the composition of the true **relative** velocities, tangent to quotient-spaces, is not

always possible. We are *not* allowed to compose the projected relative velocity of  $A$  to  $B$ , in  $B$ -space, with the projected relative velocity of  $Y$  to  $Z$  in  $Z$ -space (where  $A \neq Z$  as shown on Figure 3). Therefore the set of all projected relative velocities among all reference systems is not a group, it is a groupoid:

$$\text{Groupoid} = \{v_{AB} | \text{for all coordinate reference systems } \{t, x_A\}\}. \quad (22)$$

We believe that the theory of relativity can be formulated alternatively as a associative groupoid of relative velocity-morphisms, and not as a non-associative coset-loop parameterized in terms of abstract velocities with the unique loop-zero-velocity.

## 6 A body as a reference fluid á la Euler

The mainstream of Physics prefers an Einsteinian-like definition of relative velocity (19) in terms of scalar *space-time* measurements of events  $x_A, x_B, t \in \mathcal{F}$ .

Leonard Euler in 1754, and Minkowski in 1908 had another idea of how to represent a deformable massive body that leads to a different definition of relative velocity between reference systems. Euler's definition is not equivalent to the Einsteinian coordinate system. Instead Euler's reference system is called a monad and is given solely by *just one* vector field.

Time can not be stopped. Spacetime is like a river of events that flow into the future, a river that never can stop. The spacetime of events is an absolute physical reality that is *choice-free*. What is relative it is a three-dimensional space that always needs an artificial *choice* of some reference system as a 'center of the Universe'.

**6.1 Definition** (The Euler definition of a reference fluid). According to Euler a deformable reference fluid is defined as a vector field  $A \in \text{der } \mathcal{F}$  on space-time, that later on was baptized as the monad field. Every vector field posses an adopted coordinate system such that,

$$At \equiv 1 \quad \text{and} \quad Ax_A \equiv 0 \quad \iff \quad A = \left( \frac{\partial}{\partial t} \right)_{x_A} \in \text{der } \mathcal{F}, \quad (23)$$

$$\tau^* \in \text{alg}(\text{clocks}, \mathcal{F}), \quad A \circ \tau^* \in \text{der}(\text{clocks}, \mathcal{F}). \quad (24)$$

The velocity of the fluid is *relative*, it needs another reference system, another reference fluid. If both fluids are normalized by the same ‘absolute’ common time coordinate,  $(dt)A = 1 = (dt)B$ , then the relative velocity on spacetime is given just by the difference of vector fields as follows,

$$\mathbf{v}_{AB} \equiv \left(\frac{\partial}{\partial t}\right)_{x_B} - \left(\frac{\partial}{\partial t}\right)_{x_A} \implies \mathbf{v}_{AB}x_A = \left(\frac{\partial x_A}{\partial t}\right)_{x_B} \quad (25)$$

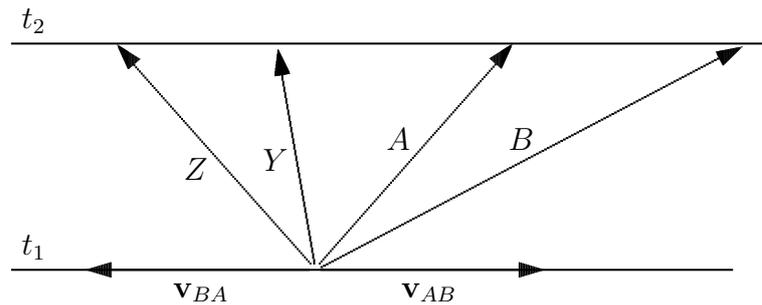


Figure 3: Four body system,  $A, B, Y, Z$ , as the future-pointed Eulerian vector fields,  $t_2 > t_1$ , within absolute Galilean simultaneity. To be in the same place is relative,  $Z$ -places are different from  $A$ -places, etc. Eulerian observer is a connection. The false relative velocities are tangent to iso-time instantaneous three-dimensional subspace, not yet projected onto relative quotient-spaces.

A coordinate  $x_S$  is an integral of motion of the street, whereas  $x_B$  is an integral of motion of the bus. The directional derivative of  $x_S$  along the street-vector-field  $S$  is zero,  $Sx_S \equiv 0$  and  $St = 1$ . Every derivative is directional, it is along some vector field. The bus  $B$ , as a vector field in a space-time, is defined by another two conditions,  $Bx_B \equiv 0$  and  $Bt \equiv 1$ . There is a conceptual difference between coordinate speed given in terms of the scalar fields (19), and Eulerian relative velocity (25) given in terms of a **vector** field. Some poor quality textbooks define a vector as a set of scalar components. However a set of scalars is just a set of scalars, and a vector does not possess scalar components if a basis is not chosen. We prefer the Euler definition of the relative velocity (on space-time) in terms of vector fields (25). This is not equivalent to the coordinate tetrad (19).

Explain connection

The central subject of the Matolcsi monograph is affine spacetime and the concept of relative velocity on spacetime. The Euler & Minkowski definition (25) above is conceptually the same as the Matolcsi definition in Part I, §6.2.2. Tamás Matolcsi in [1994, Part I, §7.1.3] is stressing

It is an important fact that the spaces of different ... observers are *different affine spaces* ...

In the present paper the relative velocity we define on a relative quotient-space without affine structure (or equivalently as a *subalgebra derivation*), and *not* on space-time as in Matolcsi Part I §6.2.2 (6.1).

## 7 Natural bundle of Lie algebroids

A surjective submersion is said to be a projection.

**7.1 Definition** (A section of space-time). Let  $\pi$  denotes a surjective submersion from four-dimensional space-time onto three-dimensional *quotient-space* (or a coset space). This surjection  $\pi$  sorts space-time-events into events in the same  $\pi$ -place, it is a fibering of space-time. An injective immersion  $s$  from a space of places into space-time of events is said to be a section for  $\pi$ , if

$$\pi \circ s = \text{id} \tag{26}$$

**7.2 Corollary.**  $\implies (s \circ \pi)^2 = s \circ \pi$  is idempotent.

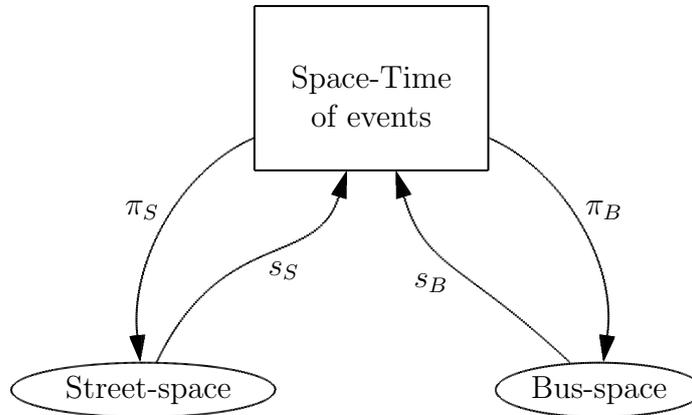


Figure 4: An injection  $s$  is a section of  $\pi$ , and a surjection  $\pi$  is a retraction of  $s$ .

Note that

$$\pi_S \circ \overbrace{s_B \circ \pi_B}^{\text{idempotent}} \circ s_S \neq \text{id}_{\text{Street-space}} . \tag{27}$$

A unital, associative and commutative  $\mathbb{R}$ -algebra (sheave of algebras) of scalar functions on space-time is denoted by  $\mathcal{F}$ . If  $\mathcal{F}$  is a commutative  $\mathbb{R}$ -algebra, then  $\text{der } \mathcal{F}$  is a Lie algebroid of derivations of this algebra  $\mathcal{F}$ , with the anchor  $= \text{id}_{\text{der } \mathcal{F}}$ . This Lie algebroid is synonymous with the Lie module, Lie-Cartan pair, Lie-Rinehart algebra, among many other names.

By  $\mathcal{F}_A$  we denote a commutative  $\mathbb{R}$ -algebra of functions on  $A$ -quotient-space. In what follows  $\mathcal{F}$  and  $\mathcal{F}_A$  extends to the Grassmann algebras of differential forms ( $\mathcal{F}$ - or  $\mathcal{F}_A$ -modules), and multivector fields.

If  $\pi_A$  is a surjection, then  $\pi_A^* \in \text{alg}(\mathcal{F}_A, \mathcal{F})$ , denotes the algebra submersion (the pull-back) of the algebra of functions and the Grassmann algebra of differential forms on  $A$ -quotient-space, into the algebra of functions  $\mathcal{F}$  and Grassmann algebra of differential forms on space-time,

$$A \in \text{der } \mathcal{F} \equiv \text{der}(\mathcal{F}, \mathcal{F}), \quad A \circ \pi_A^* \equiv 0 \in \text{der}(\mathcal{F}_A, \mathcal{F}). \quad (28)$$

We arrive to the concept of the natural bundle introduced by A. Nijenhuis [1972], with the lifting (= vertical lifting) of Grassmann algebra of differential forms, and the maps of Grassmann algebras of multivector fields. This is summarized in diagrams 5 and 6.

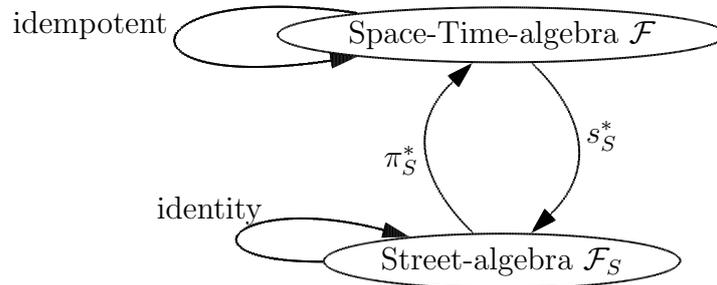


Figure 5: Maps among Grassmann algebras of differential forms: retraction and section.

Figures 4-5 demonstrate the importance of the retraction and the section (7.2), within distinct algebraic structures,

$$\begin{aligned} \text{retraction} \circ \text{section} &= \text{id}, & s^* \circ \pi^* &= \text{id}, \\ \text{section} \circ \text{retraction} &= \text{idempotent}. \end{aligned} \quad (29)$$

If  $\mathbf{v} \in \text{der } \mathcal{F}_S$  is a vector-field, a derivation, on an  $S$ -street-quotient-space, where,  $S \in \text{der } \mathcal{F}$ , is a derivation on space-time, and let  $W \in \text{der } \mathcal{F}$ , then we

have a section-dependent derivations

$$S, W \in \text{der}(\mathcal{F}, \mathcal{F}), \quad \mathbf{v} \in \text{der}(\mathcal{F}_S, \mathcal{F}_S), \quad (30)$$

$$s_S^* \in \text{alg}(\mathcal{F}, \mathcal{F}_S), \quad \pi_S^* \in \text{alg}(\mathcal{F}_S, \mathcal{F}), \quad (31)$$

$$s_S^* \circ W \quad \text{and} \quad \mathbf{v} \circ s_S^* \in \text{der}(\mathcal{F}, \mathcal{F}_S), \quad (32)$$

$$W \circ \pi_S^* \quad \text{and} \quad \pi_S^* \circ \mathbf{v} \in \text{der}(\mathcal{F}_S, \mathcal{F}),$$

$$\text{der } \mathcal{F} \ni W \xrightarrow{\text{projection}} s_S^* \circ W \circ \pi_S^* \in \text{der } \mathcal{F}_S. \quad (33)$$

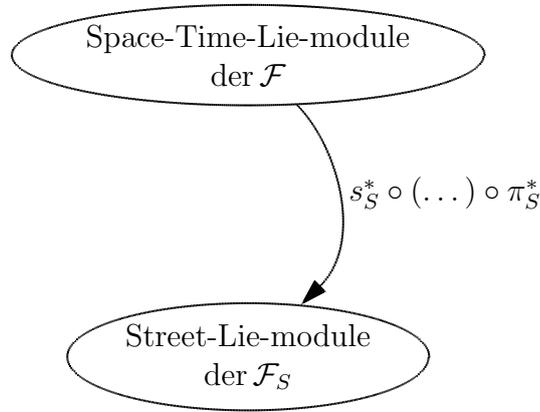


Figure 6: Lie-module map

**7.3 Proposition** (Lie algebroid/module and algebra maps). *Let  $\mathcal{F}$  be a commutative algebra, then the Lie algebroid of derivations,  $\text{der } \mathcal{F}$ , is a Lie  $\mathcal{F}$ -module. Let,  $D \in \text{der } \mathcal{F}$ . For in- and out- algebra morphisms we have*

$$\pi^* \in \text{alg}(\dots, \mathcal{F}) \quad \text{and} \quad s^* \in \text{alg}(\mathcal{F}, \dots), \quad (34)$$

$$D \circ \pi^* \in \text{der}(\dots, \mathcal{F}) \quad (35)$$

$$s^* \circ D \in \text{der}(\mathcal{F}, \dots).$$

Let  $f \in \mathcal{F}$ , thus

$$\begin{aligned} (fD) \circ \pi^* &= f \cdot (D \circ \pi^*), \\ s^* \circ (fD) &= (s^*f) \cdot (s^* \circ D). \end{aligned} \quad (36)$$

**7.4 Corollary.**  $s^* \circ (fD) \circ \pi^* = (s^*f) \cdot (s^* \circ D \circ \pi^*)$ .

## 8 Main: the relative velocity

Let  $S, B \in \text{der } \mathcal{F}$ , such that  $S \cdot B \neq 0$ , and  $S^2 \equiv S \cdot S \neq 0$ . Then

$$S \cdot \left( -\frac{B}{S \cdot B} + \frac{S}{S \cdot S} \right) = 0. \quad (37)$$

**8.1 Proposition** (The Lorentz gamma factor). *[Matolcsi 1994 2001; Bini, Carini, Jantzen 1999, 2003; Oziewicz 2005; Bolós 2006] The Lorentz gamma factor on space-time is deduced as follows. Given two observers,  $S, B \in \text{der } \mathcal{F}$ , such that  $S^2 = B^2 = -1$ , there exists the unique vector field  $\omega_{SB} \in \text{der } \mathcal{F}$ , and  $\gamma_{SB} \in \mathcal{F}$ , such that*

$$S \cdot \omega_{SB} = 0, \quad B = \gamma_{SB}(S + \omega_{SB}), \quad (38)$$

$$\implies \gamma_{SB} = -S \cdot B = \gamma_{BS}, \quad \omega_{SB} = \frac{B}{-S \cdot B} - S, \quad (39)$$

$$\gamma^2 = (S \cdot B)^2 = \frac{1}{1 - (\omega_{SB})^2} \implies \gamma_{SS} \equiv 1, \quad (40)$$

$$S \cdot B = (gS)B = -(dt_S)B = -Bt_S \implies \gamma_{AB} = Bt_S = St_B, \quad (41)$$

$$-1 = (s^* \gamma_{SB})^2 \left( -1 + \frac{\mathbf{v}^2}{c^2} \right). \quad (42)$$

Metric tensor on quotient space from metric  $g$  on spacetime,

$$g_A \equiv s_A^* g. \quad (43)$$

For every time-like vector-field  $A \in \text{der } \mathcal{F}$ , we denote by  $\mathcal{F}_A$  a  $\mathbb{R}$ -algebra of functions on  $A$ -quotient-space.

[Llosa and Soler 2004] use push-forward,  $\pi_*$  on projectable vector fields from  $\text{der } \mathcal{F}$ .

**8.2 Definition** (Main). The relative velocity vector-field tangent to  $A$ -quotient-space, a body  $A$  is observing  $B \in \text{der } \mathcal{F}$ , is defined as follows,

$$\mathbf{v}_{AB} \equiv s_A^* \circ \left( \frac{1}{-A \cdot B} B \right) \circ \pi_A^* \in \text{der } \mathcal{F}_A, \quad \text{and} \quad \mathbf{v}_{AA} = \mathbf{0}_A \in \text{der } \mathcal{F}_A. \quad (44)$$

Therefore,  $A \neq B$  imply that,  $\mathbf{v}_{AA} \neq \mathbf{v}_{BB} \in \text{der } \mathcal{F}_B$ , which is the main objective of the present paper.

In particular, if  $f \in \mathcal{F}_A$  is a function on  $A$ -quotient-space then the coordinates of the relative velocity vector are as follows

$$\begin{aligned} \mathbf{v}_{AB}(f) &= \frac{B(f \circ \pi_A)}{-A \cdot B} \circ s_A \in \mathcal{F}_A, \\ \mathbf{v}_{AB} &= \mathbf{v}_{AB}(x^i) \frac{\partial}{\partial x^i} = \left\{ \frac{B(x^i \circ \pi_A)}{-A \cdot B} \circ s_A \right\} \frac{\partial}{\partial x^i} \in \text{der } \mathcal{F}_A. \end{aligned} \quad (45)$$

These expressions, (44)-(45), show that the relative velocity depends on a free choice of  $\pi$ -section  $s$ , *i.e.*  $\mathbf{v}_{BA}(s_A)$  is time dependent. One can calculate acceleration in terms of the covariant derivative.

**8.3 Comment (Groupoid).** We arrive at the structure of a groupoid where the arrows (morphisms) of this groupoid are all relative velocities

$$\text{Morphisms} = \{\text{der } \mathcal{F}_A | A^2 = -1\}. \quad (46)$$

Objects are time-like vector-fields, all reference fluids,

$$\text{Obj} = \{A \in \text{der } \mathcal{F} | A^2 = -1\}. \quad (47)$$

We need to define the source and the target maps,

???

$$\begin{array}{ccc} \text{der } \mathcal{F}_A & \xrightarrow{\text{the source}} & A, \\ \text{der } \mathcal{F}_A \ni \mathbf{v} & \xrightarrow{\text{the target}} & B \quad \text{such that} \quad \mathbf{v} = s_A^* \circ \frac{B}{-B \cdot A} \circ \pi_A^*. \end{array} \quad (48)$$

For the target we have the following set of conditions

$$B^2 = -1, \quad B \cdot S = -\gamma_{BS} = -\frac{1}{\sqrt{1 - (\omega_{BS}/c)^2}}, \quad (49)$$

$$\omega_{SB} \in \text{der } \mathcal{F}, \quad B \cdot \omega_{SB} = \gamma \frac{\omega^2}{c^2}. \quad (50)$$

**8.4 Comment (Monad).** The reference fluid is a time-like future-pointed vector field in space-time. The Euler fluid-observer is a vector-monad, whereas the Einstein coordinate observer is a tetrad, a system of coordinates.

We have the following relation between space-time differentials, with two unknown scalar functions,  $a$  and  $b$ , both are implicitly  $\mathbf{v}_{AB}$ -dependent, where

$\mathbf{v}_{AB}$  is on space-time, not projected,

$$dx_B = a dx_A + b dt. \tag{51}$$

$$\text{Normalization: } At = Bt \equiv 1, \tag{52}$$

$$\begin{aligned} x_A \text{ is an } A\text{-space} &\iff Ax_A = 0, \\ x_B \text{ is an } B\text{-space} &\iff Bx_B = 0. \end{aligned} \tag{53}$$

Set  $v \equiv \mathbf{v}_{AB}x_A$ . This implies

$$b = -a(v)v, \quad dx_B = a(v)(dx_A - vdt), \tag{54}$$

$$a|_{v=0} = 1. \tag{55}$$

It is not obvious that the scalar  $a$  must be obligatorily  $a \equiv 1$ .

## 9 Composition of relative velocities

The composition of algebra maps,  $(\pi_S \circ s_B)^* = s_B^* \circ \pi_S^* \in \text{alg}(\mathcal{F}_S, \mathcal{F}_B)$ , it is an algebra map between quotient-spaces. It is a map from the Grassmann algebra of the  $S$ -street-quotient-space into the Grassmann algebra of the  $B$ -bus-quotient-space, see Figure 7,

$$s_S^* \circ \pi_S^* \equiv \text{id}_S, \quad \text{but} \quad s_C^* \circ \pi_S^* \neq s_C^* \circ \pi_B^* \circ s_B^* \circ \pi_S^*. \tag{56}$$

It is strange algebra map, because  $s_B^* \circ \pi_S^*$  is not invertible, cf. with (7.2),

$$s_S^* \circ \pi_B^* \circ s_B^* \circ \pi_S^* \neq \text{id}_S \tag{57}$$

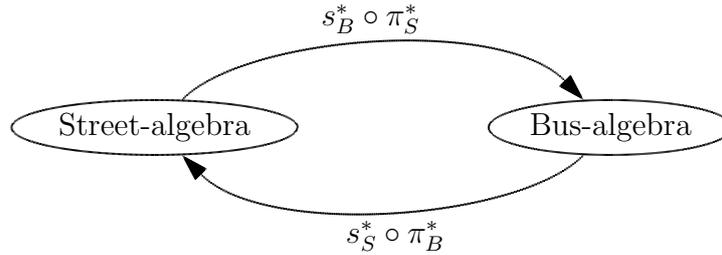


Figure 7: Galilean space is relative

In [Oziewicz 2005, 2007] we considered the composition/addition of relative velocities defined on space-time. How can we define the composition of relative velocities defined on distinct quotient-spaces?

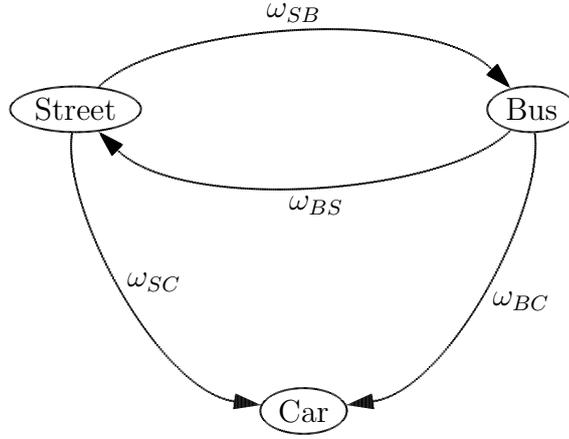


Figure 8: Relative velocities on space-time

Consider the following four algebra derivations all  $\in \text{der } \mathcal{F}$ , see Figure 8,

$$\omega_{SB}, \omega_{BS}, \omega_{BC}, \omega_{SC} \in \text{der } \mathcal{F}. \quad (58)$$

We postulated early, cf. with (??), that

$$S \cdot \omega_{S\dots} = 0, \quad B \cdot \omega_{B\dots} = 0. \quad (59)$$

We recall the expression, using metric-postulates (??),

$$s_S^*(\omega_{BC} \cdot \omega_{BS}) = (s_S^* \circ \omega_{BC} \circ \pi_S^*) \cdot (s_S^* \circ \omega_{BS} \circ \pi_S^*). \quad (60)$$

**9.1 Theorem** (Composition of velocities = ‘composition’ of derivations). *The following expression for addition/composition of relative velocities on quotient-space is deduced from the associative addition of relative velocities on space-time derived in [Oziewicz 2005],*

$$\mathbf{v}_{SC} = \mathbf{v}_{BC} \circ \mathbf{v}_{SB} = \frac{(s_S^* \gamma_{\omega_{SB}}) \mathbf{v}_{SB} + (s_S^* \circ \omega_{BC} \circ \pi_S^*)}{(s_S^* \gamma_{\omega_{SB}}) \{1 - s_S^*(\omega_{BC} \cdot \omega_{BS})/c^2\}} \quad (61)$$

We need to clarify the meaning of inverse velocity, from  $\mathbf{v}_{SS} = \mathbf{0}_S$ ,

$$(s_S^* \gamma_{\omega_{SB}}) \mathbf{v}_{SB} + (s_S^* \circ \omega_{BS} \circ \pi_S^*) = \mathbf{0}_S. \quad (62)$$

$$s_S^* \circ \omega_{BS} \circ \pi_S^* = -s_S^* \circ B \circ \pi_S^*, \quad (63)$$

$$\mathbf{v}_{SB} = (\mathbf{v}_{BS})^{-1} = +s_S^* \frac{1}{\gamma_{\omega_{SB}}} (s_S^* \circ B \circ \pi_S^*) \neq -\mathbf{v}_{BS} = +s_B^* \circ S \circ \pi_B^*. \quad (64)$$

## 10 Conclusion

We propose to consider the theory of relativity as a **groupoid** of relative velocities, and not as a group, for the following reasons.

1. We prefer to keep a separate zero-velocity for each reference system,  $A \mapsto \mathbf{v}_{AA} = \mathbf{0}_A \in \text{der } \mathcal{F}_A$ . Each reference system possesses its own zero velocity,

$$\text{der } \mathcal{F}_A \ni \mathbf{0}_A = \mathbf{v}_{AA} \neq \mathbf{v}_{BB} = \mathbf{0}_B \in \text{der } \mathcal{F}_B. \quad (65)$$

This groupoid possesses as many units=neutrals as there are different reference systems,

$$\mathbf{v}_{BA} \circ \mathbf{0}_B = \mathbf{v}_{BA} = \mathbf{0}_A \circ \mathbf{v}_{BA} \quad (66)$$

2. The order of composition is important,

$$\mathbf{v}_{AB} \circ \mathbf{v}_{BA} = \mathbf{0}_B \neq \mathbf{0}_C = \mathbf{v}_{BC} \circ \mathbf{v}_{CB} \quad (67)$$

The composition of velocities does not commute.

3. Not all velocities-morphisms are allowed to be composed.

**10.1 Comment** (Groupoid versus group). Every relative velocity tangent to quotient-space, Definition 8.2, has an inverse,

$$\begin{aligned} \mathbf{v}_{BS} &= \left( s_B^* \frac{1}{-B \cdot S} \right) \cdot (s_B^* \circ S \circ \pi_B^*) \in \text{der } \mathcal{F}_B, \\ (\mathbf{v}_{BS})^{-1} = \mathbf{v}_{SB} &= \left( s_S^* \frac{1}{-S \cdot B} \right) \cdot (s_S^* \circ B \circ \pi_S^*) \in \text{der } \mathcal{F}_S. \end{aligned} \quad (68)$$

But not all velocities tangent to quotient spaces, and **indexed** by source-system and target-system, can be composed.

If we consider the velocities  $\mathbf{v}$  and  $\mathbf{u}$  just in the abstract space of ‘velocities’ then it looks like every pair of velocities is composable and therefore we have a group. However if  $\mathbf{v}$  is concretely,  $\mathbf{v}_{AB} \in \text{der } \mathcal{F}_A$ , a velocity of a body  $B$  relative to  $A$  (then  $\mathbf{v}_{BA} \in \text{der } \mathcal{F}_B$  is relative to the source-body  $B$ , the minus sign  $\mathbf{v}_{BA} = -\mathbf{v}_{AB}$  is misleading), and similarly  $\mathbf{u}$  is concretely  $\mathbf{u}_{YZ}$ , as shown on Figure 3, then the composition of these velocities,  $\mathbf{v}_{BA} \circ \mathbf{v}_{YZ}$ , can not be interpreted as a relative velocity in the case that  $A \neq Y$ .

The coset loop of velocities was uncovered by Abraham Ungar in 1988, we refer to Ungar's new book [Ungar 2010]. However there is a rival alternative. If the composition of the velocity of the Sun relative to the Earth  $\mathbf{v}_{ES}$ , with the velocity of Mars relative to Jupiter  $\mathbf{v}_{JM}$ , is not a relative velocity, we are no longer within the relativity group with non-associative coset loop of velocities. We are within an associative relativity groupoid.

## A Velocity as element of the coset loop

The relativity group, both the Galilean group and the Lorentz group, define the concept of an abstract velocity  $v$ , as a group-parameter,  $\text{Group} \xrightarrow{v} \mathbb{R}^3$ , to which one can refer without the qualifying phrase 'relative to'. These relativity groups are six-dimensional, every element is parameterized by three parameters of some rotation subgroup, and three components of the velocity. The velocity belongs to a coset space,  $\mathbf{v} \in G/R$ , where  $R$  is a some rotation subgroup of  $G$ ,  $R < G$ . In the case when  $G$  is the Galilean group, every rotation subgroup is a normal subgroup,  $R \triangleleft G$ . Whereas a rotation subgroup in the Lorentz group is not a normal subgroup,  $R \not\triangleleft G$ . If  $G$  is the Galilean group then the coset space of group velocities,  $G/R$ , is a group with associative addition of velocities. If  $G$  is the Lorentz isometry group of symmetry of invertible space-time metric tensor, then the coset Lobachevski space of all velocities  $G/R$  inherits a loop structure (unital quasigroup) also known as a non-associative group. This implies that the addition of velocities within the Lorentz relativity group is non-associative and it is strange that this simple fact was realized 80 years after the foundations of special relativity were established [Ungar 1988]. See also recent Ungar's monograph [Ungar 2010].

Our proposal about the concept of relative velocities can be summarized as follows: the associative groupoid of all relative velocities is a rival theory of the non-associative coset loop.

## B Tetrad is a basis

Every coordinate system is a system of scalar fields that determine a coordinate co-frame of exact differential forms, and a dual coordinate-frame

of vector fields given by partial derivatives,

$$\{x^\mu\} \implies \{dx^\mu\} \iff \left\{ \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \right\}, \quad (69)$$

$$dx^1 \wedge dx^2 \wedge \dots \neq 0. \quad (70)$$

Therefore every coordinate system gives a particular basis in a module of differential one-forms, and a basis in the module of vector fields. These bases in fact are coordinate-free, expressed in any other coordinate systems are the same, *i.e.* coordinate-free,

$$dx^\mu = \sum \frac{\partial x^\mu}{\partial y^\nu} dy^\nu. \quad (71)$$

The only peculiarity of the coordinate bases is that every one-form in a co-frame is exact. When this differential condition is relaxed we arrive to the concept of a moving-frame introduced by Cartan. A moving co-frame is a basis of differential one-forms and a dual basis of the vector fields

$$\{\alpha^1, \alpha^2, \dots\} \iff \{X_1, X_2, \dots\}, \quad \alpha^\mu X_\nu = \delta_\nu^\mu, \quad (72)$$

$$\alpha_1 \wedge \alpha_2 \wedge \dots \neq 0, \quad d\alpha^\mu = \text{not necessarily zero bi-forms}. \quad (73)$$

## C What is a material body?

**C.1 Definition** (Reference system). Every *reference system* is a classical macroscopic material body. We avoid the word ‘frame’ because we consider frame to be a synonym of basis in some vector space, and a **reference system** need not to be the same as a *reference frame/basis*.

Each basis determine unique metric tensor, such that it is orthonormal and diagonal in this basis. Therefore metric tensors can be seen as elements of a coset  $GL/\text{Lorentz}$  group, although we prefer to see metric tensor as basis-free.

Metric connection are supposed to describe *massless* gravity field, therefore one can ask where is a massive body within tetrad formalism?

$$\text{tetrad} \implies \text{metric} \implies \text{connection} \implies \text{gravity} \quad (74)$$

The adequacy of the Einsteinian concept of coordinate scalar fields as a material body was criticized by Léon Brillouin in the monograph ‘Relativity

Reexamined' published by Brillouin's wife in 1970, after Brillouin's death. Chapter 4 of Brillouin's monograph is entitled *A badly needed distinction between mathematical sets of coordinates and physical frames of reference*. Brillouin asked where is there a mass in the coordinate system?

The usual statement of the relativity principle requires that frames of reference be extremely heavy.

...

A frame of reference does not constitute a piece of unreal geometry anymore; it is a heavy laboratory, built on a rigid body of tremendous mass, ...

Léon Brillouin, 1970, §4.3, page 45

We refer also to Brillouin's Section 4.5 on page 49, entitled *Sets of mathematical coordinates or physical frames*,

When reading Einstein's papers, one can readily see that he does not make this distinction and ascribes to sets of coordinates (without mass) properties that apply only to heavy frames of reference. But first let us discover in those very papers a premonition of the *frames of reference* (Albert Einstein, *Annals of Physik* **35** (1911) page 898). In the second section of this paper, Einstein writes:

Let us consider two **material** systems  $S_1, S_2 \dots$

Léon Brillouin, 1970, §4.5, page 49

The first condition in (5),  $Vt = (dt)V = 1$ , is no more then the artificial convention of simultaneity that gives 'normalization' of the reference fluid,

$$t = \text{clock} \circ (\text{simultaneity relation}). \quad (75)$$

The normalization (75) can be generalized to  $V$ -transversal differential one-form  $\tau_V$ ,  $\tau_V V \equiv 1$ , that not need to be neither exact nor closed  $d\tau_V \neq 0$ . Let  $g$  denote the Minkowski metric tensor. The Minkowski proper-time differential one-form is convention-free but metric-dependent,

$$\begin{aligned} \text{der } \mathcal{F} \ni V &\xrightarrow{g} \text{proper-time one-form} = gV, \\ V^2 \equiv g(V \otimes V) &= (gV)V = -1 \in \mathcal{F}, \quad \text{i.e. } gV \simeq -dt. \end{aligned} \quad (76)$$

The Pfaff differential one-form,  $gV \in (\text{der } \mathcal{F})^*$ , - the Minkowski  $g$ -dependent proper-time<sup>2</sup> - introduced Minkowski in his Appendix in 1908 - need not to be exact.

**C.2 Definition** (Inertial mass). The reference-dependence of velocity implies the reference-dependence of energy. Every reference system possess its own energy relative to exactly that one system, and this own energy is precisely an inertial mass of a body. We define inertial mass as energy of the zero-velocity.

The energy-momentum contents of the above reference fluid  $V \in \text{der } \mathcal{F}$ , with the inertial mass-density,  $\rho_V$  (and with a pressure for a non-viscous perfect fluid) is encoded in a differential Pfaff-form on space-time, a time-like 4-momenta  $p_V$  proportional to proper-time one-form  $gV$ ,

$$p_V \equiv -\rho_V c^2 gV \quad \text{and} \quad p_V^2 = -\rho_V^2 c^4. \quad (77)$$

Energy of a body  $V$  as seen by reference fluid  $U$  is denoted by  $E_{UV}$ ,

$$E_{UV} = p_V U = -\rho_V c^2 U \cdot V = \rho_V c^2 \gamma, \quad E_{VV} = p_V V = \rho_V c^2. \quad (78)$$

Almost equivalently, a scalar inertial mass density is given by the trace of the energy-momentum tensor  $T$ , see e. g. [Tolman 1934, 1987 §86],

$$T \equiv V \otimes p \quad \Longrightarrow \quad \text{trace } T = pV = \rho c^2 \quad \text{and} \quad T^2 = \rho c^2 \cdot T. \quad (79)$$

## D Center of mass

Let  $0 < m_X$  be the mass of body  $X$ , and  $0 < m_Y$  be the mass of body  $Y$ . The Galilean center of mass of  $X$  and  $Y$  is the convex sum  $Z$  (analogous to the convex sum of mixed states in quantum mechanics),

$$Z = \left( \frac{\partial}{\partial t} \right)_z = \frac{m_X \left( \frac{\partial}{\partial t} \right)_x + m_Y \left( \frac{\partial}{\partial t} \right)_y}{m_X + m_Y} \quad (80)$$

---

<sup>2</sup>Often special relativity is interpreted superficially as a theory where time is relative, that time is dilated. Special relativity allows the metric-dependent proper-time and a multitude of metric-free non-proper-times related to the conventions of simultaneity. Only Minkowski metric-dependent proper-time is dilated.

It is nontrivial to find a coordinate function  $z$  describing the center-of-mass the bodies without some extra assumptions. However under a doubtful extra axiom we have

$$\left(\frac{\partial x}{\partial t}\right)_y = \left(\frac{\partial y}{\partial t}\right)_x \implies z = m_X x - m_Y y. \quad (81)$$

The commutative symbol of addition ‘+’ must be used for addition of relative momenta on space-time (and *not* for composition of velocities on relative spaces!),  $\mathbf{p}_{AB} \equiv E_{AB}\mathbf{v}_{AB}$ , when the source-body  $A$  is the same,

$$\mathbf{p}_{AX} + \mathbf{p}_{AY} = \mathbf{p}_{AZ}. \quad (82)$$

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