The Geometric Concept of Number

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The development of the real number system represents both a milestone and a cornerstone in the foundation of modern mathematics. We go further and suggest that the real number system should be completed to include the concept of direction. Some of this work has already been done by the invention of the complex numbers, quaternions, and vectors. What has been lacking, however, is a general geometric number system. In 1878, William Kingdom Clifford invented his "geometric algebra", based upon the earlier work of Grassmann and Hamilton. Geometric algebra is the completion of the real number system to include new anticommuting square roots of plus and minus one, each such root representing an orthogonal direction in successively higher dimensions. All of the usual rules of the real number system remain valid, except that the commutative law of multiplication is no longer universally valid. The book, "New Foundations in Mathematics: The Geometric Concept of Number" in preparation by the author, represents an attempt to show at an undergraduate level how many ideas of modern mathematics can be developed within this new framework, including modular number systems, complex and hyperbolic numbers, geometric algebra of Euclidean and pseudo-Euclidean spaces, linear and multilinear algebra, Hermitian inner product spaces, the theory of special relativity, representations of the symmetric group, calculus and differential geometry of n-dimensional surfaces, Lie groups and Lie algebras, and other topics.

1. What is Geometric Algebra?

Geometric algebra is the completion of the real number system to include new anticommuting square roots of plus and minus one, each such root representing an orthogonal direction in successively higher dimensions. If one new square root $i = \sqrt{-1}$ is included, we have the complex numbers. If, instead, one new square root $u = \sqrt{+1}$ is included, we have the hyperbolic numbers [1]. The complex and hyperbolic number systems are commutative number systems, $ab = ba$ for all complex or hyperbolic numbers a, b. However, the hyperbolic numbers have new interesting properties unseen in the real or complex number systems. For example, if we define $u_+ = (1+u)/2$ and $u_- = (1-u)/2$, we find that $u_+^2 = u_+$, $u_-^2 = u_-$, $u_+ + u_- = 1$, and $u_+ u_- = 0$. We say that u_+ and u_- are mutually annihilating idempotents, which partition unity. The hyperbolic number plane has many analogous properties to the complex number plane, replacing the unit circle with the 4-branched unit hyperbola.

If three new anticommuting square roots

$$i = \sqrt{-1}, \quad j = \sqrt{-1}, \quad k = \sqrt{-1} \tag{1}$$

are introduced into the real number system, we arrive at Hamilton's famous quaternions. Note that aside from the fact that the new square roots are anticommutative, all of the usual algebraic properties of the real numbers remain valid. In the case of the hyperbolic numbers, we do have the existence of zero divisors, $u_+ u_- = 0$, when neither of the factors is equal to zero. In general, the geometric product of geometric numbers obeys exactly the same algebraic rules as the addition and multiplication of square matrices of real numbers. Indeed, this is no coincidence. For each geometric algebra, one can find a corresponding algebra of square real matrices which have identical algebraic properties. The great advantage of geometric algebra is that the elements of the geometric algebra have an unambiguous geometric interpretation which is sadly lacking in a square matrix of real numbers.

2. The Inner and Outer Products

To keep things simple, let us briefly explore the geometric algebra G_3 of the three dimensional Euclidean space of our physical World. Starting with the real number system, we invent three new anticommuting square roots e_1, e_2, e_3 of +1 which we choose to represent orthogonal unit vectors along the x, y, z axes of the Euclidean space R^3. Thus, $e_1^2 = e_2^2 = e_3^2 = 1$ and $e_{ij} = e_i e_j = -e_j e_i = -e_{ji}$ for $1 \le i < j \le 3$. The e_{ij} are given the geometric interpretation of unit bivectors in the respective *xy*, *yz*, and *xz* planes. Still to be accounted for is the unit trivector $I = e_{123} = e_1 e_2 e_3$, which represents the 3-dimensional direction of an oriented unit cube. The geometric numbers of space are pictured in Fig. 1.

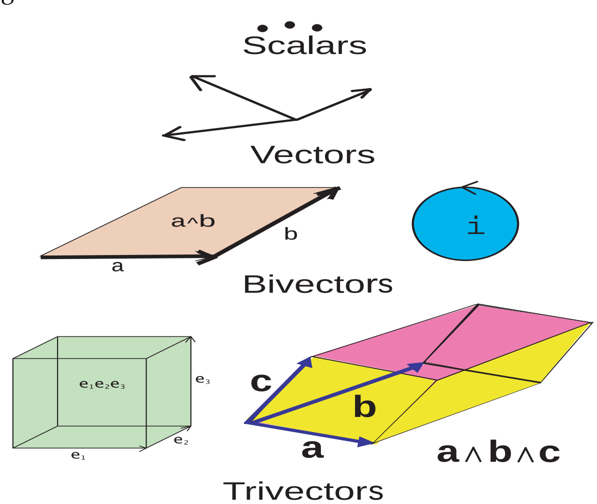

Fig. 1. Geometric Numbers of Space

Up until now we have only discussed the geometric product, its geometrical interpretation, and its algebraic properties. We now discuss the inner and outer products which are defined in terms of the geometric product.

Let $a=\sum a_i e_i$, $b=\sum b_i e_i$, $c=\sum c_i e_i$, where the sums are over $i=1,2,3$, and the a_i, b_i, c_i are real numbers, be three vectors in R^3. The *inner product* of a and b is defined by $a\cdot b=\frac{1}{2}(ab+ba)=a_1b_1+a_2b_2+a_3b_3$, and the outer product is defined by $a\wedge b=\frac{1}{2}(ab-ba)=Ia\times b$, where $a\times b$ is the ordinary *cross product* of Gibbs-Heaviside vector algebra. The formulas for the inner and outer products follow from the rules obeyed by the basis vectors e_1,e_2,e_3 given above, and should be derived by the serious reader. From the definitions given above, it is clear that the geometric product of two vectors satisfies the basic identity $ab=a\cdot b+a\wedge b$. It is this unification of the inner and outer products that give the geometric product the power that neither the inner nor the outer products have separately.

Probably the most difficult part in mastering geometric algebra is coming to grips with the definitions and identities satisfied by the inner and outer products of geometric numbers. Nothing in this World is free, and the same thing is true for acquiring language and mathematical skills. Let us examine one more algebraic identity regarding the relationships between the inner and outer products. We have

$$a\cdot(b\wedge c)=\frac{1}{2}\left[a(b\wedge c)-(b\wedge c)a\right]=-a\times(b\times c)$$

and
$$a\wedge(b\wedge c)=\frac{1}{2}\left[a(b\wedge c)+(b\wedge c)a\right]=I\left[a\cdot(b\times c)\right]$$

For the convenience of the reader, we have expressed the inner and outer products of a vector and a bivector in terms of the better known Gibbs-Heaviside triple products. Once again we find that $a(b\wedge c)=a\cdot(b\wedge c)+a\wedge(b\wedge c)$, so the geometric product of a vector and a bivector is the sum of the inner and outer products of the vector a with the bivector $b\wedge c$.

3. Geometric Calculus

There are many geometric algebras, but all are obtained by introducing new *anticommuting* square roots $\sqrt{\pm 1}$ into the real number system [2, 3]. Geometric algebra is deeply connected to linear and multilinear algebra, and indeed the subjects should be developed together. Every finite dimensional geometric algebra $G_{p,q}$ is the geometric algebra of a *quadratic form* of signature p,q, with p the number of anticommuting unit basis vectors which have square +1, and q the number of anticommuting unit basis vectors which have square -1. The geometric algebra of spacetime $G_{1,3}$ is generated by one *time-like vector* $e_1{}^2=1$, and three *space-like vectors* $e_2{}^2=e_3{}^2=e_4{}^2=-1$ [4]. Each geometric algebra provides the algebraic framework for the development of a geometric calculus and differential geometry [2]. The fundamental theorem of calculus for a 2-dimensional surface S in the Euclidean space R^3, with bounding curve C, when represented in the geometric algebra G_3, takes the form

$$\int_S g\,dx_{(2)}\partial_x f=\int_C g\,dxf$$

and includes Green's and Stokes' Theorems.

4. Conclusion

Geometric algebra offers new geometric tools for the study of vector calculus and differential geometry, representation theory of Lie algebras and Lie groups and many areas of mathematics, physics, and robotics. The completion of the real number system to include the concept of direction provides a powerful geometric number system which is the foundation of a lot higher mathematics where geometric concepts are involved. As David Hestenes told me as a graduate student in mathematics at Arizona State University in the 1960's,

> *"Algebra without geometry is blind, geometry without algebra is dumb".*

The reader is referred to [5], where many additional links to geometric algebra and its applications can be found. In addition, [6] gives a link to my talk given to World Scientific Database which provided the basis for this paper. I would like to thank the organizers of World Scientific Database for inviting me to contribute to these proceedings.

References

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