Remarks on Physics as Number Theory

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There are numerous indications that Physics, at its foundations, is algebraic Number Theory, starting with solid state physics evidence in the context of the universal model of Quantum Computing and Digital World Theory. Bohr's Model for the Hydrogen atom is the starting point of a quantum computing model on serial-parallel graphs is provided as the quantum system affording the partition function of the Riemann Gas / Primon model. The propagator of the corresponding discrete Path Integral formalism is a fermionic Riemann zeta function value "closely" related to the experimental value of the fine structure constant of QED. The Klein geometry of the primary finite fields unravels a rich structure of the set of prime numbers, and logic reasoning as well as quark masses lead to the conjecture that Fermat primes correspond to quarks.

1. Introduction

Plato thought reality is based on numbers; this predates the current trend in physics that Number Theory is the ultimate Physics, the same way the atomist theory of Democritus predates at a conceptual level the modern theory of atoms.

This article will explain at a conceptual level how the fine structure constant is related with a zeta value, starting from a model of physical realty based on a categorification of number theory[1].

From these considerations we are led to remarks about the Dirichlet "quantum" duality in number theory as an underlying core of Riemann hypothesis; and then, on to the "hidden" hierarchic structure of the (multiplicatively) prime numbers.

Although some remarks are reported as implicit questions to the specialist, the author hopes that the reader will enjoy the sketched picture with its underlying riddles, and perhaps help with comments.

2. The Electromagnetic Coupling Constant

Before we embark in a number theoretical excursion to understand the *fine structure constant* **α**, a brief research background should be sketched.

2.1. Non-mainstream Research Provides New Clues

Let us start with a brief recall of some recent emergent ideas in physics should be in order.

- **Clue 1:** No Ambient Space-Time!

There is no *underlying* (ambient) space-time background, conform with the philosophy of *General Relativity* that all is energy-matter (energy-momentum tensor) defining the interaction and its strength, the metric.

Note that back then, at the beginning of 20th century Einstein did not have the adequate mathematics tools "to do" geometry with representations of graphs, in the spirit of Feynman, or like in Kontsevich deformation quantization theorem and Formality Theory. In fact, since "all" is discrete, a good-enough model is an algebra of graph representations [16] [2].

- **Clue 2:** *Non-linear* Electromagnetism

The SU_2-symmetry of electro-weak theory is not broken, since clearly by now EM is not a U(1)-theory, and Maxwell's theory, as truncated by Heaviside and others, is just an approximation of quantum information waves (non-commutative *U(2)*-theory) unifying the Maxwell-Hertz type of waves (transversal) with the Meyel-Tesla type of waves (longitudinal) [40].

- **Clue 3:** There are no particles, nor fields!

There are no elementary "particles", since the wave-particle duality reflects the fundamental dichotomy between local and global properties, which is technically formulated as the Classical/Quantum Computing hierarchy ((Master-Slave, in Computer Science jargon, or mathematically: homology-cohomology duality theory). In particular the separation "particle-field" is no longer appropriate to a theory accepting that there is no "in between particles empty space".

The categorical point of view, which includes the basic principle that morphisms modeling relations between objects are more important, and in fact define the objects themselves, is the appropriate language for the modern physics of interactions, as a new science altogether then Newton-Einstein classical physics of motion.

So, in the author's opinion, a good model is the above mentioned representation of graphs, implementing "supersymmetry" not by joining the external to the internal symmetry groups, but by unifying fermions as the objects and bosons as the morphisms. *SU(2)* (and *SL(2,C)*) as coefficients play the role of (local) Lorentz transformations, as well as the role of a connection between local "pieces of space-time" (qubits), in the spirit of Heisenberg's Matrix Mechanics[3]. The correspondence is the well known Hermitian (or Klein) correspondence used also in Penrose's twistor program [2], [3]. It will be briefly referred to as *2+2*=3+1* correspondence (spinors versus quaternions).

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[1] Additional details and formulas can be found in [1].

[2] All you can do with Poisson manifolds can be done better with differential graded coalgebras of graphs modeling the interaction structure, "The Matrix", and with Lie algebra coefficients as local symmetries; it is the QFT upgrade of lattice gauge theory.

[3] This quantum computing model ahead of its time, without the support of Classical Computing, was destined to lose the battle against the traditional differential equations approach, leading to recession in physics: the Schrodinger's Equation.

On the experimental side, particle accelerators as a tool of investigation, clearly favors particle aspects and mislead to the idea that quarks are particles, yet forever confined, since the computational physics ("Don't think; just compute!") leads to a lack of balance between an experimental physics "a la Galileo" and the conceptual insight of Greeks and oriental philosophy. Even worse, this leads to the "certainty" that the electron is "point-wise", just because in these experiments it does not reveal any "parts", hence "no internal structure" making it a point-wise object.

... So, what *is* then!? ...

2.2. The Quantum Information Paradigm

Everything there is or becomes, is quantum information processing: qubits/quantum registers and quantum gates. The Universe, at "machine code" level is "just" I/O-processes between systems, as an *SU(2)*-analog of Markov processes[4].

These are just Feynman diagrams without the embedding in an ambient space-time, but rather decorated in the above mentioned Formality Theorem [16]. The moments associated to the diagrams are replaced by qubits, as "pieces of relativistic space-time"[2].

A long story short, QID is in a sense an upgrade of *Theory of Electric Circuits*, which is known to be an approximation of *Maxwell's Electro-Magnetism*[5], once we change coefficients from U(1) to U(2), and propagating "chunks of space-time" through gates, i.e. qubits/quantum registers, while allowing the network to fluctuate via insertion-elimination of subgraphs.

Similar to GR, which models dynamics as free motion in curved space vs. Newton's curved motion (force) in free space [5], QID is an "Eulerian formulation" where "space" moves and "matter" (local info/structure) stays and "witnesses" the changes.

2.3. Multiply-connectedness and Quantum Mechanics

The new main feature of such a theory is multiply-connectedness, since now space is a network of interconnected subsystems.

And QM is just a way to deal with this; the mother of all quantum experiments (Feynman) has a simple explanation: there is no "particle"' traveling from the source through the double slit to the screen, and in the loop formed there is a resonance of two spinors, as initiated by Feynman-Wheeler theory, or the more recent process of hand-shake of Cramer's Transactional Interpretation of QM, and bi-states approach of Aharonov-Vaidman's.

A resonant standing wave is formed, which satisfies the properties longitudinal waves of Tesla [40], as presented by K. Meyl; the "beat" of the two spinors propagating in opposite directions, transfers energy-momentum, and at the I/O ports of measurement, corresponds to the "photon" (never at rest, but with a velocity which can be decreased to 38 mph, as shown in experiments with several concurrent lasers).

[4] At a much higher level (for us), it is a 3-dimensional "Game of Life", with five senses as controllers; I wonder how does the world appear to a cell: a sort of "Flatland", or "radial"!?

[5] There are authors sustaining the correspondence between EM and QED through Dirac's equation, using quaternions, via the Hermitian correspondence, the bridge 2+2*=3+1.

So the quantum classical concept of photon is appropriate only at Input-Output ports of the system (predominantly an input/ source: lasers; predominantly an output/ target: detectors).

The proposed model resolves the half a century old conceptual stumbling block in QM, which Feynman was warning us about during 70s ("Nobody really *understands* quantum mechanics!").

But how did QM deal with this at the beginning? Heisenberg's quantum mechanics represents a way to reduce representations of graphs and the corresponding dynamics which appears as "stochastic process", in terms of functions, by using superpositions.

A bipartite graph (I/O-graph), as a relation, is a list of functions (special relations); with weights attached to edges/arrows, probabilities or amplitudes, it becomes a matrix: so, "back to functions"!

So, the "mystery" of QM evaporates, once a relational approach is used; mathematically this amounts to treating cofunctions on an equal footing as functions.

2.4. Fundamental Theorem of Arithmetic and Serial-Parallel Graphs

The above emerging "integer" quantum computing is a study of resonant modes of graphs; the basic configurations are the serial-parallel graphs, thought of as a bouquet of paths, replacing the traditional circle picture of the Bohr's model of atom.

Mathematically speaking, the Boolean analog of this interpretation of the fundamental theorem of finite Abelian groups, is the normal form of Boolean functions, associated to classical logic.

Factorization of integers viewed as "shadows" of the objects Z_n, corresponds to serial and parallel composition in the category of finite Abelian groups: $n=\prod p_i^{e_i}$.

Space-like or parallel addition is the direct sum $Z_n X Z_m$ (multiplication), and time-like or serial composition is the semi-direct product for p-groups:

$$Z/p^eZ=(Z/pZ)\ \alpha\ (Z/pZ)\ \alpha\ ...\ \alpha\ (Z/pZ).$$

It models harmonics k=1..e (finite range!) of a fundamental frequency 1/p, related to the so called *Primon Model*.

In this model one just "propagate" integers (mathematical quanta), which is an "absolute" way to deal with probability theory (counting quanta n_i, leading to a "conservation law" in a L^2-theory: Hilbert geometry, and conformal quantum mechanics; not the relative ratios n_i/N, of an L^1-theory: measure theory).

We claim that this *Quantum Computing Model of Interactions*, when applied to simple system, like the hydrogen atom, is a better model of the H-atom spectrum then *Bohr's model* with its energy levels:

$$\frac{E_n}{E_0}=a\frac{1}{n^2},\qquad E_0=m_ec^2 \qquad [1]$$

It is definitely needed in QID as a grand unifying theory, modeling the electron and proton together as a Hopf fibration [2]. This is essentially the underlying reason for the electro-weak theory to "work" (not to mention the supersymmetric approach using representations of graphs - with emphasis on monodromy rather than connections / gauge theory or metric theories -, unifying "particles" and "waves", i.e. fermions and bosons).

2.5. The "Old" Quantum Mechanics: Bohr, Sommerfeld, de Broglie

It is worth emphasizing that the "game" of interest here is "Physics Models Design". When we talk about "electron", "photon" etc., we implicitly refer to a theory as a context.

Bohr's model for the H-atom implies a mechanical Newtonian model (particle side of the picture), together with a quantization condition hinting to a wave model, which later was generalized by Sommerfeld into the celebrated Bohr-Sommerfeld quantization conditions (BS-Quantization: $\int pdq=nh$) for the bound system.

These early version of particle-wave duality was later extended to the unbounded systems by de Broglie, who introduced the pilot-wave concept, while emphasizing that it is fictitious, and not a material wave.

This theory amounts to a *direct correspondence* between mechanic variables of the Newton/Einstein particle model and the variables of the wave model, in or outside bound systems (resonant cavities/ systems with feedback), conform to the celebrated *principle of correspondence* and wave-particle duality, at conceptual level. This "old quantum mechanics" turns out to be the right way to go ([10], string theory, QID, etc.), leading to the mathematics of periods on graphs, Riemann surfaces etc. (quantum numbers as topological invariants, symplectic quantization and Maslov index etc.).

The step back via Schrodinger Equation of Wave Mechanics, although leading to a more familiar territory[6], is reminiscent of the "simplification" of Maxwell's equations achieved by Heaviside, removing its most important concept, namely the vector potential as irrelevant[7]. Even gauge theory matured slowly until its importance was partially reestablished by the Aharonov-Bohm work.

To rise back at Maxwell's level, who considered the vector potential as a physically meaningful quantity, would mean to introduce the ether and "upset" Einstein; eliminating the concept of background space-time avoids this conflict, and solving many more problems (including String Theory's "Landscape problem", the continuum multitude of Feynman Paths etc.).

The consequence: coordinates do not have a physical intrinsic meaning, and result only from a path integration of momentum; correspondingly, there are no "pointwise" charges, but only integrals of current, and periods (electric charge: 2 period, and magnetic charge: 1-period).

In conclusion, extending Bohr's circular model for the H-atom to a quantum computing model on Serial-Parallel graphs, PoSet for the Path Integral formalism, is a discrete modern version of "old quantum mechanics", sort of a Phoenix Bird of Physics.

2.6. Fine Structure Constant and the Propagator

Now the "fine structure" constant α is determined by what the *propagator* in the corresponding Theory of Path Integrals is, and as apparent in the above formula for energy levels [10, 11], is not due to relativistic corrections.

The H-atom itself is a "piece of space" (*2+2*=3+1*, see [2]), so it is relativistic to start with.

So, let's not consider the Bohr's model of the Hydrogen atom in isolation, as a mini-solar system, since we target modeling interactions, but two H-atoms exchanging a quantum of energy as a whole. Now, instead of modeling separately the subsystems as a pair of fermions exchanging a boson as in particle-field formalism, let's model the system as a whole, "a la" string theory, but as a "categorical dumbbell"; geometrically, since the Hopf bundle is a torus fibration, think of it as a genus two Riemann surface:

Fig. 1. From QED to Riemann Surfaces via duality.

2.7. Electro-Magnetic Charges and α=e2/hc

Now electric and magnetic charges (fluxons) are 1-periods (cohomology with some coefficients, here taken to be integers), representing impedances (type L for magnetic, and type C for electric):

$$Hall: g_m = h/e, \qquad Coulomb: g_e = e/c.$$

Regarding charges, it is important to understand their role as sources and sinks of (generalized) energy-momentum [2] (Analysis of Lorentz Force: the work part defines the electric force, and the curvature/monodromy part, the magnetic force; it is related to relative Hodge Theory).

Electric charge represents a non-zero divergence (2-period in the Maxwell-Heaviside EM, or 1-period with *SU(2)* coefficients in QID: the discrete 2nd Green Theorem), and magnetic charge represents a source of generalized momentum (1-period / fluxon: the discrete 1st Green Theorem) [2].

In the QID model (think EC for simplicity), they are related with the unit of action (Plank's constant), by Hodge decomposition:

$$\frac{1}{c}\cdot h = g_m \cdot g_e \longleftrightarrow Z_0 \cup Z_3 = Z_1 \cup Z_2$$

The ratio g_m/g_e for the "fine structure" constant (propagator) is related to the average frequency of the SP-graphs/Riemann Surfaces:

$$1/c^2 = \varepsilon\mu \longleftrightarrow 1/\omega^2 = C\cdot L \longleftrightarrow \alpha = g_m/g_e,$$

$$1/\omega^2 = \frac{1}{2\pi}T^2{}_{harmonic\_av.} \longleftrightarrow \varsigma(2) = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

This zeta value ζ(2)=π²/3! is in fact related to the partition function of the *Riemann gas* (*Primon Model*), with the "size" of the Lie algebra of primitive elements of *Z*(the primes). Its inverse is the fermionic propagator (fermionic zeta function value):

$$\varsigma^- = 1/\varsigma = \sum \mu(n)/n^2$$

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[6] Heisenberg's I/O-automaton approach, later developed into the S-matrix theory, is a too early quantum computing with amplitudes, appearing before even a theory of classical computing was established!

[7] Progress, as any other action-reaction, seems to exhibit the oscillatory behavior of a mass-spring system.

representing the probability of two numbers (composite particles) to be relatively prime (non-correlated). Here the Mobius function μ is the convolution inverse of the constant function 1.

Compared with the Bohr's H-atom spectrum formula Eq. [1], it gives an interpretation of mechanical rest mass energy $E_0=mc^2$ as being of "EM origin" (vibration origin/ inductance):

$$\sum E_n / E_0 = a \cdot \varsigma(2).$$

Recall that in the spirit of General Relativity, the mass coefficient (mechanics) or inductance coefficient (EM), together with the Hook's elastic force coefficient (mechanics) or Coulomb's charge (EM), represent coefficients of the metric (via Hodge structure):

$$Ds^2 = \varepsilon dr^2 - \mu^{-1} dt^2 \longleftrightarrow * = diag[\mu^{-1}, \varepsilon],$$

when interpreted as Lagrange equations:

$$m(d\dot{x}/dt) + R\dot{x} + k\int \dot{x} = 0, \quad m = \frac{\partial^2 L}{\partial \dot{x}^2},$$

provided the Lagrangian *L* is non-degenerate (*R* damping constant or electric resistance).

This interpretation should be considered in the context of the model *Γ->M*, where we embed the vibrating network *Γ* into a mechanical relativistic space-time *M* (recall the generalized momentum *P=mv+eA* and ether interpretation).

2.8. Path Integrals and Propagator

Since we don't know much about moduli space of Riemann Surfaces $M_{g=2}$, we wi'll compute the correlator in the discrete version (the "real" case any ways, since the circle should be viewed as the limit $S^1=\lim Z_n$; conformal geometry is in fact S^1-equivariant theory, and quantum physics is projective precisely because it is a theory of periods: "topological" quantum numbers.

The corelator in a PoSet (or finite category with a path integral) [7] is given by the the Mobious function [22].

For example [7], when the integers Z ("hiding" the category $\{Z_n\}$), is regarded as the PoSet {n->n+1}, the finite difference (Df)(n)=f(n)-f(n-1) is representable as the convolution with Mobius function μ, which is the inverse of the constant arithmetic function u:Z->C, u(n)=1, which then appears as the associated fundamental solution:

$$Df=\mu * f, \quad \mu * 1=\delta,\ Du=\delta,\ \int f(n)dn=\sum f(k)=f*1.$$

Indeed, this follows from the Fundamental Theorem of Calculus (Stokes Theorem) $\sum=D^{-1}$ and the corresponding convolution inverse formula $\mu=u^{*-1}$, which provides the corresponding Green function (fundamental solution of Du=δ, where δ is the identity for convolution/path integral).

Now since the convolution * is here (POSets) a Path Integral (and in a more general framework of groupoids, e.g. Feynman processes [9]) the Green function is the propagator, i.e. the kernel of the Path Integral in the sense of Feynman.

Let us also note that in the Dirac formalism, the integer analog of Dirac operator D=√Δ is represented by the Mobius function μ as a Dirac kernel. Now its Dirichlet series ζ_-=DS(μ) (the fermionic zeta function) satisfies the following super-symmetric decomposition of the partition function:

$$\zeta_- \zeta_+ = 1,\ \zeta_+ = DS(1),\ \zeta_- = DS(\mu).$$

with Riemann zeta function $\zeta=\zeta_+$ the *bosonic partition function* (trivial character 1, respectively $\mu=(-1)^{sign}$ [12]).

This facts are just shadows of statements in *Integer CFT* via categorification, with its Serial-Parallel Structure Theorem and its shadow, the Fundamental Theorem of Finite Abelian groups, briefly mentioned above (see also [12], for ideas of development of *Integer CFT*, and connections with *Rational CFT*).

In our context of SP-graphs, the associated probability *P* of two SP-graphs to be relatively prime *(n,m)=1* (2-point correlation function, i.e. the propagator), is the probability of an integer affine vector in the Gauss plane to be a generator (defines a symmetry, i.e. an automorphism).

Now it is a standard fact that this probability is a (fermionic) zeta value [21]:

$$P = P(\gcd(n,m)=1) = \frac{1}{\sum 1/n^2} = 1/\varsigma(2) = \varsigma_-(2).$$

Now we interpret 1/n as possible exchanges of quanta (configurations) between the two H-atoms, as reflected by the Bohr's spectrum above. In our model unifying particles and fields, it is an interaction cobordism, playing the role of space-time joining two Cauchy hypersurfaces.

Now "energy = log(probability)", so a relation between α and log ζ_- is expected. In [8] the "coincidence":

$$-\frac{2}{\log\alpha} \approx \sum_{p\in SpecZ} \frac{1}{p^2}$$

relating *log(a)* and a value of the *prime zeta function* was noticed from a totally different perspective (a relation between the masses of quarks and prime numbers!).

If we use Euler's formula for the zeta function, we find:

$$\sum_{p\in SpecZ} \frac{1}{p^2} <-> \sum_p \log(1-\frac{1}{p^2}) = \log(1/\varsigma(2)) = \log(P).$$

Then, are quarks a "quantum"/multiplicative version of the prime modes, i.e. is QCD an exponentiated version of EM? Does it mean that QCD is a non-linear theory, based on multiplicative number theory vs. the usual linear formalism based on superpositions and additive number theory?

2.9. "Fundamental" Constants are Anthropomorphic Values

At this stage it appears that there are NO fundamental constants: they all are related, and computable from the combinatory of number theory. Plank's constant reflects the discreetness of action, Einstein's constant *c* reflects a limitation of capacity of information processing (space-to-time conversion; impedance $c^2=1/\varepsilon\mu$ etc.); electric and magnetic charge are (relative) Hodge dual periods with relative ration (again an impedance) the fine structure constant, which is the propagator in the Integral CFT of finite abelian groups $C(\{Z_n\},+,X)$. It is the categorification of *Z*, the fundamental Hopf ring, and a model of replacing the field of fractions, which are sufficient in the commutative setup (left=right fractions), and needed to implement division, with divisibility theory using comultiplication as a natural counterpart of multiplication.

Here the direct sum + is the monoidal operation, and multiplication X is the second operation in the direction of the dimension of depth of structure, i.e. the associated p-adic norm exponentiates the degree of hierarchy of structure of the system (higher frequences correspond to sub-processes).

So, loosely speaking, everything is harmony (standing waves and exchange of info) or the lack of it (I/O-ports and transfer, when impedances don't "match "), all expressible in terms of integers as periods ("integrals" over cycles).

It will turn out that "prime numbers" are not structure-less and there is a hidden structure expressible in terms of symmetries. If this later structure has a "space"-like conotation (structure of internal symmetries; quark types and apparent grouping into generations), the p-adic dimension has a "time"-like conotation ("processing frequency"; harmonic analysis and atomic spectra).

3. Primon Model for the Hydrogen Atom

3.1. Quantum Mechanics: A Few Points of History

The 1900's quantum physics models were, in succession: Bohr's model, improved by Somerfeld, Heisenberg Matric mechanics taken over by Schrodinger's formalism, followed by a return to quantum computing via Feynman Path Integral.

Since QC (and Heisenberg Matrix Mechanics) is essentially about resonance modes of graphs as "spinotronics" circuits (similar to RLC-circuit theory), and since Bohr-Sommerfeld quantization conditions are a primitive way to introduce homological periods into the formulation of quantum theory, we will "upgrade" Bohr's model of the Hydrogen model to the Serial-parallel Quantum Circuit Model; its mathematical-physics implementation is the Primon Model [14] (Riemann gas).

To a resonance line (absorption/emission capable) from the spectrum of the H-atom (Hopf bundle), from which we consider for now only the principal quantum number *n*, is interpreted as a Serial-Parallel graph associated with the prime factorization of *n*.

For example, for n=45, the quiver associated to the factorisation of the primon model state $3^2 \cdot 5$ is:

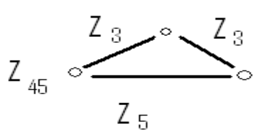

Fig. 2.

with composition of morphisms representing the prime factors of 3 correspond to the semi-direct product Z_3 X_r Z_3, while in the spatial direction it is the direct sum (monoidal product in the category of finite abelian groups).

In view of the Bohr-atom model, the serial-parallel decomposition should rather be pictured as a bouquet of closed paths of length the exponents of the corresponding primes (1-chains in the Lie algebra of primitive elements *P*; we will refer to this, briefly as a bouquet of circles).

Conceptually, we implement the resonances not as Bohr-Somerfeld quantization conditions due to de Broglie waves forming a standing wave on circular Bohr orbits of an imaginary mini-solar system, but rather as proper modes associated to the corresponding SP-graph, as an automaton modeling the "black-box" we call Hydrogen atom.

In fact we proceed to a reverse engineering of the H-atom; what impedances (LC corresponding to permitivities μ and ε as Hodge structure), will yield on such a SP-graph the frequency whose multiplicative Dirichlet series decomposition (see later § Duality) is the prime factorization of *n*?

In other words, we refine the Fourier analysis of the H-atom as a generator of a pure spectrum, which corresponds to the well-known energies above the rest mass, by "exponentiation" to multiplicative theory of quantum amplitudes and probabilities. We emphasize that this approach binds the energetic picture E_n of spectral lines, to the probabilistic picture, with their amplitudes (winding numbers/periods etc.) in a discrete context of a complexified Markov stochastic process:

Exp: Bohr-Sommerfeld Periods -> Heisenberg Mechanics;

Lie algebra/ Primitives -> Lie groups/FPI on groupoids.

Now an edge of the SP-graph has a conductivity/impedance of $z_e=1/p$, where p is a prime number. It can be argued that "conductivity", as the inverse of resistance which is a metric parameter, is in fact introduced by a p-adic norm.

Irrespective of the underlying reason that z_e corresponds to a primitive element in Z (integers), it follows that that the corresponding partition function is the Riemann zeta function [14].

Mode directly, as explained above, when counting the probability of "interaction" of two such SP-graphs, the zeta value at β=2 (inverse of kT) is obtained:

$$1/Z_{eff} = \Pr obability(SP(n), SP(m)) = 1) = \sum \frac{1}{n^2} = \varsigma(2) \cdot$$

Here Z_{eff} is the *effective impedance* (harmonic average, via Möbius inversion).

3.2. Quantum Boolean Forms for Graphs

Now, the SP-graphs are a sort of a canonical "Boolean form" for graphs, in the sense of the analog statement for classical computing, namely that a Boolean function (classical logic/computing) can be expressed canonically in a disjunctive form as a conjunction of elementary products of Boolean variables (see Wiki).

Is there a similar canonical form representation for graphs?

More explicitly, a graphs has cycles (homology) and, with a Hodge structure corresponding to the impedances of its edges (RLC-elements), it will have periods representing the proper modes of resonance (spectrum; see [15]).

These modes ω_k should correspond to the cycles Zp^e in a factorization theorem for the graphs which reduces to the Fundamental Theorem for Finite Abelian groups when graphs are cycles.

3.3. Fermionic Riemann Function and Adeles

Since primes are "fermionic" (graded Lie "algebra" of primitive elements P=Spec(Z)[1], in view of applying Milnor-Moore Theorem etc.), one should compare it with the fermionic zeta function, which is the Dirichlet Series of the Mobius function:

$$\varsigma_- = DS(\mu), \mu(n) = (-1)^{sign(n)}, \quad \varsigma_- \cdot \varsigma_+ = 1, \varsigma_+ = \varsigma \, .$$

Then its logarithm, via Euler formula, is sort of a Fourier equation involving the p-adic norms (adelic numbers):

$$\log Z_{eff} = -\log \zeta_- = -\sum \log(1-\frac{1}{p^2}) \approx$$

$$\sum \frac{1}{p^2} = \| \sum_{p \in Prim(RQ)} p \|^2_{Adelic} = \| Prim(CQ) \|^2_{Adelic}.$$

This is also the partition function of the primons, the primes as generators of N (or of the complex group algebra of (Q_+,x)), at temperature β=1/2, corresponding to the primary energy levels $E_p = \log p$.

On the other hand, the "vacuum energy" should be:

$$(mc)^2 : \sum (E_p / c)^2 = h^2 \sum \nu_p = h^2 \| Prim(CQ) \|.$$

3.4. Summary

In conclusion we propose the Quantum Computing Model of the Hydrogen atom of QID. The proton and electron are modeled together in a SU_2-formulation of EM, as the quantum system for the *Primon Model* (Riemann gas) having the Riemann zeta function as its partition function. This model "upgrades" Bohr's Model for the H-atom once viewed as a "Solar system", together with Bohr-Sommerfeld quantization conditions based on de Broglie hypothesis, to a Quantum Computing Model (SP-graphs with prime impedance - to be understood later as hiding a hierarchic structure) together with a spectrum of proper modes of resonance, corresponding to RLC-circuit theory from the physics side and Hodge periods from the mathematics side. This ends the dark ages of physics [41] following the dead end of QM based on wave functions of Schrodinger's equation, with its ad-hoc collapse; it is a quantum reprogramming as advocated by J. Post, and compatible with the main theories outside main-stream research (Wheeler-Feynman, Creimer, Aharonov-Vaidman etc.). The Bohmian approach, with its intricate order, is the song of the swan of a wave mechanics supposed to die long before the famous abominable cat.

Recall that the "rest", i.e. "perturbative" theory, is just a dg-coalgebra technical game (Feynman graphs cohomology [16], [9], where graphs are like "digits" of quantum precision, graded by the number of "loops" (cycles): inserting and eliminating graphs via graph extensions (see also Connes-Kreimer algebraic renormalization).

Moreover, the S^1-equivariant cohomology (complex projective spaces <-> conformal theory: winding numbers etc.) is replaced by discrete covering spaces Z/nZ, in order to eliminate the continuum. This leads directly to Galois Theory, instead of first reverting from Schrodinger Equation (and wave function formalism, since there is no ambient space, really), and to Path Integrals on groupoids (there is a well known correspondence between Schrodinger Eq. formalism and FPI), and then to monodromy theory (see [13]; also a common trend: from gauge theory to Loop Quantum Gravity etc.), where solutions of DE are replaced by branched covers, and in the discrete realm by Path Integrals on groupoids (see [7] - Discrete calculus).

Finally, as seen above, the QC SP-graph model allows deriving the EM coupling constant (fine structure constant) as a propagator (probability of "interaction", as a cobordism).

4. Duality and Riemann Hypothesis

Duality is an overarching theme in mathematics (Poincare, Pontrjagin, Fourier, Tanaka-Krein, categories with duality) and physics (particle-wave duality, CPT-Theorem etc.).

Dirichlet Transform comes from a duality at the level of multiplicative number theory. The "facts" belong to two levels: categorical level, and Number Theory (their shadows).

The braided category C(Z) = *{Z/nZ}* of finite spaces of moduli (also denoted Z_n) has the Hopf ring *Z* as its Grothendieck ring (left/right orientations in Z_n correspond to duality).

The Fundamental Th. of Finite Abelian groups, with its shadow the F. Th. of Arithmetic, allows to define the exponential

$$\exp: g \to (N, \bullet, 1), \qquad g = ZP[1]$$

between the free module of primitives with basis the prime numbers *P=Spec(Z)*, where additive number theory lives ("Lie algebra", generators/ "infinitesimal level"), and the Hopf monoid of natural numbers *(N, ·,1,Δ,η)* of multiplicative number theory ("Lie group", symmetries/groupoid level).

The orientation ± sign was omitted for simplicity. The Lie algebra g is graded, for later use with Milnor-Moore Theorem, and physics interpretation as a fermionic algebra.

A partition of $k=k_1+...+k_1$ (invariant factors) represents the coefficients of a g-chain. Exponentiated, yields a "quantum number" (SP-graph):

$$n = \exp\left(k_1 p_1 + \ldots k_1 p_1\right) = \prod p_i^{k_i}, \qquad Z_n = \otimes_{i=1..l} \propto_1^{k_i} Z_{pi}.$$

Recall that the p-groups $Z_p{}^k$ are group extensions 2-cocycle the carry-over unit. It is a generalized "clock", where the "p-minute" has *p* seconds, the p-adic "hour" has *p* minutes etc.

Unlike Fourier transform, there are finitely many harmonics in such a multi-hierarchic object (quantum Fourier transform is "compact").

Rather than thinking of the subgroups of $Z_p{}^k$ as time-like / frequency harmonic analysis, one should interpret it as a hierarchy of structure, in the sense of Haar wavelets theory (details correspond to higher frequency, and therefore lower "de Broglie wave-length").

Now there is a duality (evaluation) between the Hopf monoid *N* and the algebra of *C*-valued functions *F(N)*:

$$<,>: \Im(N) \times N \to C$$

from the multiplicative Number Theory, multiplicative functions are of interest: *(n,m)=1 => f(mn)=f(m)f(n)*. These functions are determined by the values on powers of primes (tensor functors).

Most notable multiplicative functions satisfy a 2-cocycle condition. For example the *Euler totient function* (number of symmetries; or the "loop functor" *Aut(Zn)*:

$$\Phi(n) = |Aut(Z_n)|, \qquad \Phi(n)/n = \prod_{p \varepsilon Supp(Log(n))} (p-1)/p,$$

or better the "probability" $P(n) = \Phi(n)/n$ (density of symmetries; "charge" density), from which it differs by a *completely multiplicative* "uninteresting" function *I(n)=n*, satisfies (on irreducible factors):

$$\delta P(p^k, p^1) = P(p^k)P(p^1)P(p^{k+1})^{-1} = 1 - \frac{1}{p}.$$

4.1. Application to Primon Model

Now δP is multiplicative (independent of the exponents k and l), and allows to define energy levels. Compare with the energy assignment $E_n=E_0 \log p$ in the Primon Model.

Theorem 4.1 *Let the (total) Hamiltonian of the Primon Model be $H_{int}=Log(\delta P)$ as a 2-chain potential between SP-states. Then it is exact, and there is a potential energy spectrum E(n), determined as an integral over the common support ("space"), with Radon-Nicodym derivative p(p):=1/p representing the density of sources, i.e. the density of charges.*

Proof. Some clarifications are in order. The interaction Hamiltonian should be, in an Einsteinian spirit, a curvature, i.e. a 2-cocycle (Riemann-Hilbert action etc.): $H(n,m)$. It is natural to be determined by the "local symmetries", i.e. a density of symmetry function, especially in the light of Noether Theorem, where charges correspond to symmetries, and via a Coulomb-Poisson formalism, would determine the potential energy (harmonic potential and general Yang-Mills equations etc.).

The spectrum of "energy levels", is really an additive 2-cochain $H(n,m)$ of work $\int F dr$ on the transition path from the state n to the state m (do not assume a kinetic/potential separation). The states are SP-graphs, which could be interpreted as cobordisms ("space-times"), where the primes, corresponding to independent processes (resonance modes), constitute the "space" and the depth of hierarchy of structure constitutes the "time" (frequency) dimension.

If the Hamiltonian 2-chain is exact (so path independence implies that a potential energy can be defined), it will depend only on the difference n-m, and a ladder of energy levels (with a "ground level", e.g. zero energy at infinity) can be defined.

So, the stipulated definition:

$$H(n,m) = -Log(\delta P(n,m)),$$

is quite mandatory. Then, because δP is multiplicative (it is an exact cocycle), it defines an energy via a Riemann-Stieltjes integral (discrete). So, the energy levels, as "jumps" at "atomic" sets, are:

$$(n,m) = p^k \Rightarrow p(p^k) = \delta P(n,m) = -\log(1-1/p) = 1/p$$

Here the logarithm is the discrete logarithm, corresponding to the harmonic potential $1/n$ (see 3.1)

The theorem gives a justification of the ad-hoc choice for the energy levels of the Primon Model (Riemann Gas):

$$E_p / E_0 = \int_l^p p(x)dx = \int_l^p 1/x dx = \log p,$$

except that a consistent use of discrete-algebraic tools (no limits!) demands a replacement of the natural base e and the corresponding logarithm:

$$E_p / E_0 = -\log(1/p) = -\log(1-(1-1/p)) =\approx -(1-\frac{1}{p}) = \frac{1}{p} - 1$$

4.2. Probability and Information/Energy

At this stage we recall that the logarithm of a probability has the meaning of information content, with its expectation value the Shannon entropy (see also [17]):

$$I = -\log P, H_s =< I >= \sum P_k I(P_k) = \sum P_k \log P_k .$$

Since our probability $P(n) = |Aut(Z_n)|/|Z_n|$ is manifestly an information content (Shannon entropy and Boltzmann entropy): $S = \log|Aut(System)|$, we will denote it as such:

$$Info: \quad I(n) = -\log P(n), \quad I(n) = \sum_{p|n} \log(1-1/p) = \sum_{p|n} 1/p$$

In other words our "charges", as *sources*, are the symmetries $Aut(Z_n)$, which correspond to generators (the elements relative prime with n, $U_n \equiv Aut(Z_n)$; recall $\Phi(n) = |U_n|$); therefore the above density of charge is the information content at a conceptual level, not only as a ("fine-tuned") formula: $p(p) = I(p)$. For the prime modes the information content is: $I(p)=1/p$. Again we compare with the ad-hoc definition of the energy levels of the Primon Model:

$$(E_p + E_0) / E_0 = 1 + \log p \approx I(p).$$

To further ponder on these aspects, recall that the groupoid formulation of the Feynman path Integral demands the inclusion of a symmetry factor (see [16, 17]):

$$A(I,O) = \int_{\tau \varepsilon Hom(I,O)} \frac{\exp(-S(\tau))}{|Aut(\tau)|}.$$

Interpreting symmetries as an information content, allows to include information on an equal footing with energy in the action (Hamiltonian or Lagrangian):

$$I = \log|Aut(\cdot)|, \quad S_{Tot} = S + I, \quad A\int \exp(-S_{Tot}).$$

Recently, it has been demonstrated experimentally that information can be converted into energy (via an analog of Maxwell's demon). It was already known that erasing information requires energy (Landauer's Principle).

These are just additional "clues" that numbers and symmetries (classical information) together with (homological) duality (quantum information) should provide the foundations of physics formulation of reality.

4.3. Energy, Mass and Coupling Constant

Assuming Einstein's formula $E=mc^2$ and the generalized formula for the H-atom energy levels:

Conjecture 4.1 *In the POSet / Path Integral formalism of the Primon Model, the coupling constant α, as a Mobius propagator, is related to the energy levels $E_n=H(m+n,m)$ as in the generalized H-atom spectrum formula:*

$$\frac{E(n)}{E_0} = \alpha \frac{1}{n^2},$$

where $E_0 = m_0c^2$ is the rest energy, with m_0 the rest mass of the corresponding Lagrangian model (Euler-lagrange equation, with mass as a metric coefficient).

This yields the value of the coupling constant of the Primon Model:

Conjecture 4.2 *The "fine structure" constant of the Primon Model is $\zeta_-(2)$ and the rest mass of the charge is related to the prime zeta value $P\zeta(2)$.*

Proof. From above, the propagator is $P((n,m)=1)=\zeta\_(2)$. The total energy is:

$$I_{tot}^2 = \sum I(p)^2 = \sum 1/p^2 = P\zeta(2).$$

Note that energy and momentum appear together in the relativistic picture, since the Hamiltonian does not necessarily split into kinetic and potential energy. Now think of mass as a density (of symmetries/charges), related to probability (via energy) as in $E=mc^2$. Then, if $\sum E_n / E_0 = \alpha\zeta(2)$ (conjecture) then:

$$\log(\sum E_n) - \log(E_0) = \log(\alpha) + \log(\zeta)2)).$$

Since (|7|) $1/\log\sqrt{\alpha} \approx \log\zeta\_(2)$, it remains to relate the probabilistic L^1-norm (non-relativistic) picture and the L^2-norm relativistic / quantum mechanics picture.

Now at this stage, we recall a possible connection between gravitational constant, which may normalize rest mass, and α [3]

$$\tau = \frac{Gm_e^2}{k_C e^2} \approx 10^{-43} \approx \exp(-1/\alpha).$$

In other words, if normalizing both Coulomb's Law for EM and Newton's Law for gravity so that $k_C = 1$ and G = 1, then the charge over mass ratio appears related to the EM coupling constant:

$$m_e / e \approx \exp(-1/\alpha)$$

Here the "charge" is information, with density $p(p) = 1/p$, and "rest energy" is E_0.

Notice that the H-atom energy spectrum formula seams additive / non-relativistic.

Also the significance of Bohr H-atom energy levels as cumulative energies, differ in interpretation with the Primon Model energy levels $E_p=\log p$ and our interpretation $E_p=1/p$; the relations between them are not yet clear at this time.

4.4. Dirichlet Transform

Returning to the "quantum" duality between $G=(N,\cdot,1,\Delta,\eta)$ and multiplicative arithmetic functions *F(G)* (compare with the context of Milner-Moore Theorem: Lie algebra, Lie Group, UEA and convolution group/function algebra):

$$<,>: \Im(N)\times N \to C$$

notice that the role of dual group of characters $\hat{G}$ is played by the completely multiplicative functions indexed by complex numbers *C:*

$$\Phi_s(n) = n^s, n\varepsilon N, s\varepsilon C.$$

Recall that a completely multiplicative function is characterized by its values on primes (vector exponent notation):

$$f(\prod p^e) = \prod f(p)^e, \quad f(p) = P^{\beta_p}, \beta_p = \log f(p)/\log p.$$

Now if β_p ("temperature") is equivariant $\beta_p = s$ (constant) then the f(n) = n^s. The transform associated with the above duality is known as the Dirichlet series:

$$(DTf)(s) = \sum f(n)\Phi_s(n) =< f,\Phi_s >.$$

For example the Riemann zeta functions, fermionic and bosonic, are such transforms:

$$\zeta\_ = DT(\mu), \quad \zeta = \zeta_+ = DT(1)$$

As expected (compare with Fourier Transform), DT is a morphism from the convolution algebra $(\Im,*,\delta)$ with Dirichlet convolution *

$$f*g = \cdot\circ(f\otimes g)\Delta, \quad \Delta(n) = \sum_{dd'=n}(d,d'),$$

and convolution unit Dirac's delta function supported at the neutral element $\delta(n) = \delta_n^1$:

Theorem 4.2

$$DT(f*g) = DT(f)\cdot DT(g), \quad DT(\delta) = 1$$

Since the Mobius function is the convolution inverse of the constant function 1 (neutral element in the algebra of complex function)

$$\mu * 1 = \delta \Rightarrow DT(\mu)\cdot DT(1) = 1,$$

i.e. the fermionic/bosonic partition functions (see Riemann Gas and [12] are inverse to one another: $\zeta\_\cdot\zeta_+ = 1$.

4.5. On the Riemann Hypothesis

Rather than saying that the zeros of the Riemann zeta function are located at *Re=1/2*, one can equivalently say that the *poles* of fermionic partition function $\zeta\_ = DT(\mu)$ have the form

$$s_k = 1/2 + i\sigma_k.$$

In fact this is a statement about the equivariant characters, which at primitive elements take the value

$$\Phi_{s_k}(p) = p^{1/2+i\sigma_k} = \sqrt{p}\cdot\exp(i\xi_k), \xi_k = \sigma_k \log p$$

Now there should be a connection with Diophantine equations, Galois extensions

$$L = Z[i][\sqrt{p}] \ni \alpha + \sqrt{p}b, a,b \in C$$

and, of course elliptic functions. The products of purely "fiber elements" $\sqrt{p}$ a $e^{i\xi_q}$ at (prime) roots of unity are reminiscent of the poles of a *Weierstrass function* for the elliptic function:

$$P(z_1;z_2) = \sum_{Q\in Lattice \quad L} V(z_1+Q) - V(z_2+Q), V(z) = 1/z^2.$$

If we think of a string theory interpretation, then one could speculate that the "string" Z_q (roots of unity ξ_q) propagates in the Diofantine torus $Z[\sqrt{p}](g=1)$, with proper modes (resonances: poles) related to the Weierstrass "potential function".

4.6. Inverse Dirichlet Transform

Perron's formula allows to recover the arithmetic function *f(n)* from its Dirichlet Transform [22]

$$F(s) = DT(f) = \sum f(n)\exp(s\gamma_n), \qquad \gamma_n = \log n$$

yet via contour integration (complex analysis), and not in a direct way from the discrete duality itself:

$$A(x) = \sum_{1..[x]} f(n), \qquad A(x) = -\frac{1}{2\pi i}\int_{c-i\infty}^{x+i\infty} F(s)\exp(xs)\frac{ds}{s},$$

where $\sum'$ represents the modification of the step function (discrete anti-derivative of $f(n)$:

$$(D^{-1}f)(n) = \sum_1^n f(k)$$

by redefining the values at discontinuities as averages of the left and right value, a phenomenon familiar from Fourier analysis. The correction uses the discrete derivative of *f*:

$$(Df)(n) = f(n) - f(n-l), \qquad A(x) = D^{-1}f(|x|) - \frac{1}{2}Df(\delta_z(x)).$$

In the discrete realm $\frac{dA}{dn}$ is essentially *f*, so Perron's formula is an intertwining property:

$$D^{-1}f(x) = 1/2\pi \int\int_{c-i\infty}^{c+i\infty} F(s)\exp(ixs)dsdx.$$

Then the residues of the poles of the DT determine the coefficients of the arithmetic function. Here complex analysis should be thought of as *U(1)*-equivariant cohomology theory (conformal geometry and winding numbers), and the discrete analog as a sort of Z_n -equivariant theory, with a correspondence due to $U(1)=\lim Z_n$.

From the physics point of view λ_n are energy levels, i.e. form the spectrum of a hermitian operator, and the above transform is an expansion into eigenfunction, probably a discrete analog of a Laplacian (e.g. Sturm-Liouville Theory etc.).

4.7. Number of Zeroes of Riemann Zeta Function

Another hint that we should look at zeroes as periods, comes from the formula for the number of zeroes of the Riemann zeta function (poles of the fermionic analog) [23, p. 47]:

$$N(2\pi T) = T\log T - T + O(\log T) = \int_0^T \log x dx + O(\log T).$$

With the usual notation $p_n = 1/2 + i\sigma_n$, one should look for the meaning of normalized zeros:

$$\gamma_n = \frac{1}{2\pi}\sigma_n, \qquad \xi_n = \gamma_n \cdot \log n.$$

In view of the "Lie algebra" of primes as primitive elements, σ_n should be an average of an inner product:

$$\sigma_n = \frac{\sum_{p|n} k_p \tau_p}{\sum_{p|n} k_p \log p},$$

where τ_p should be some periods, or "quantum numbers" in the Riemann gas (Primon model). If $P_i = exp(E_1/kT')$ are probabilities for the states X_p corresponding to the prime number p_i, of energy $E_i = logp_i$, then $d\log n = \sum k_i dE_i$, and

$$N(2\pi N) = \int_0^N \log x dx \sim \sum_0^N \log n = \sum_{Exp(k\cdot X_p)\le N} \langle k, \log p\rangle \cdot$$

In other words, the zeros of the Riemann zeta function seem to be linear combinations of some periods for a homological basis corresponding to prime numbers (Jacobian matrix). This is consistent with their association with the eigenvalues of a hermitian operator [29, p. 12] (Here the normalization $/2\pi$ was included in γ_n):

$$\sigma_n = (\gamma_{n+1} - \gamma_n)\log\gamma_n.$$

The interpretation of the zeros as *moments* (periods etc.) suggests its comparison with Shannon (Boltzmann) entropy:

$$S_N = \sum \gamma_n \log \gamma_n, \qquad \delta_n \sim S_{n+1} - S_n = DS(n)$$

in the context of the partition function for the Riemann Gas (the usual Lagrange multipliers maximization relating the energy levels and temperature).

The Feynmann Path Integral like formula for the long-term dependency between zeros [29, p. 13] suggests that:

$$A = \sum_{i=N+1}^{N+40000} \exp(i\sigma_n x),$$

represents an amplitude, to be related with the Mobius / Discrete calculus alluded above (SP-graphs etc.). A "non-linear harmonic analysis" frequencies / energies of $x = log(p^q)$, reveals spikes (resonant / proper modes) when the exponent q is a prime number

$$A = \sum_{i=N+1}^{N+40000} \exp(2\pi i q\xi_n) = \sum_{i=N+1}^{N+40000} p^{2\pi i q\gamma_n}.$$

As pointed out by M. Berry [30], this is the behavior of the eigenvalues of a quantum system with a Hamiltonian, which is not random, without time-reversal symmetry (e.g. hierarchy instead of time?), with chaotic phase-space trajectories. This quantum chaos is the result of embedding say SP-graphs in an ambient space-time, since one *must* quantize the resulting "classical system ("undoing" this "classic-ization"), to render coordinates and momenta as non-physical at Heisenberg-Planck scale)

4.8. Classical vs. Quantum Fourier Transform

The parallel between Fourier transform (S^1-harmonic analysis and duality) with the theory of Dirichlet characters is well known (e.g. [25, 26]).

The circle $S^1 = R/Z$ is the "moduli space of the lattice $Z \to R$ with corresponding projection. Here *Z* (the discrete realm) is the additive group which is better thought of as the *linear PoSet* with its Path Integral (Discrete Calculus of finite differences).

On the other hand $\{Z_n\}_{n\in Z}$, as a category, which reflects into the finite/quantum Fourier transform, is better viewed (enriched) as the category of Serial-Parallel graphs; or at the global level of *Z* as a Hopf ring, as the PoSet of divisibility (multiplicative number theory: Hopf monoid $(N,\cdot,1,\Delta,\delta_1)$ and its dual of arithmetic functions *Hom(N,C))*.

The relation between the two is established by exponential, or correspondence between generators/primitive elements (S^1 is commutaive, with one generator; Z seems commutative - see below the consequences of the embedding $Aut(Z_n)\to S_{n-1}$, with *P=Spec(Z)* generators). The following correspondence is essential in establishing the link between the additive and multiplicative theories:

$$U_n = Aut(Z_n) \cong Z_{\Phi(n)} \to S^1.$$

The above "picture" should be "doubled", i.e. extended to the S^1, and Z_n-equivariant theory level, by considering the complex plane *C = R[i]* and Gauss plane *Z[i]*. Then the exponential and other branched covers (and deck transformations) should be viewed in the light of the theory of Riemann surfaces.

4.9. Approximations via Complex Analysis

A widespread "habit" in number theory is to approximate / estimate number theoretical functions using complex analysis; asymptotic formulas reflect the limit $S^1 = \lim Z_n; e.g. \Phi(x) \sim x$ is the approximation of the "exact" formula [26, p. 14]:

$$\psi_0(x) = x - \frac{\xi'(0)}{\xi(0)} - \sum_{poles \in \xi_-} \frac{x^{r_k}}{r_k}.$$

But Perron's formula is an *intertwining formula* (see above), comparing a discrete concept (Dirichlet transform) with a continuum concept (Cauchy evaluation of residues of poles). Differentiating both sides, essentially yields a relation between f and $\hat{F}$ (shifted):

$$f(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\exp(ins)ds.$$

Since $c = 1/2$ is special, it would be interesting to study the "real" version of Dirichlet transform *f*(t) = DT(f)(1/2+it)* and its relation with Fourier transform in the above formula (compare with Laplace transform, when boundary or initial values are present).

The "exact" formulas should refer to the Hopf duality mentioned above, without introducing the limit $S^1=\lim Zn$. Then the connection with the distribution of primes in Z (continuum case, via Perron's formula [18]), would probably involve the "homological" basis of primes (Milnor-Moore Theorem etc.), leading to a "multiplicative estimate" ($\exp: ZP \to N$). To better understand this, it is natural to look deeper into the structure of the prime numbers. There should be no surprise that the (pre-Lie algebra of) rooted trees enter the scene: it's a Path Integration and Hopf algebra "game", like in the Hopf algebra approach to renormalization of QFT (see the articles by Connes and Kreimer).

5. The Hidden Structure of the Prime Numbers

Prime numbers look simple as a shadow of finite abelian groups Z_p. in fact, the multiplication of integers cannot distinguish between the serial and parallel composition ($\oplus$ and $\propto$) in the braided category *Z*.

If we look at Z_n as spaces of a Klein geometry under the action of $Aut(Z_n)$, then the primes reveal an interesting structure:

$$Z_p: \quad (Aut(Z_p),\circ) \equiv Z_{p-1} \equiv \prod q_k^{e_k}.$$

For example, Z_{13} has an orbit-action structure due to the factorization of its symmetry group into $Z_2{}^2$ and Z_3.

Definition 5.1: The *hierarchy tree* of a prime number p has nodes powers of 2 and descendents corresponding to the factorization of its symmetry group. It is constructed inductively as follows.

If $p-1 = 2^k q_1^{e_1} \cdots q_1^{e_1}$ then the node is labeled by k (classic binary information) and the descendents are the trees associated with the primes q_j with multiplicity e_j.

For example, let us reconstruct the number associated to the tree 2 - -(2- -2, 1, 1) (the multiplicity 2 for the second leaf was written explicitly).

First, the subtree - -2 yields the prime $2^2 + 1 = 5$ as the first descendent of the big tree. The other prime (with multiplicity 2, is $2^1 + 1 = 3$. Finally, we have $n = 2^2 \cdot 5 \cdot 3 \cdot 3 + 1 = 181$, which happens to be prime. This is not always the case:

$$2^3 \cdot 7^2 + 1 = 393 = 3 \cdot 313,$$

not a prime.

Definition 5.2: *The hierarchy trees of prime numbers are called Fermat Trees.*

For example, Marsenne primes have trees which are Fermat trees with one node 2^n.

There are plenty of questions emerging: 1) what hierarchy trees yield primes? 2) How often such trees yield primes? 3) Is it often enough to threaten RSA? etc. The above defined trees generalize the *proth* primes [13], which are primes of the form $2^k \cdot n + 1$, by considering the whole tree structure instead of the top level. Such a hierarchy is similar to the one present in the "continuous fraction" representation, or Haar analysis etc. (graded structure). Some additional examples can be found at [24].

The primality of numbers of the form $N = 2^k \cdot n + 1$ can be tested in terms of the Jacobi symbol [32, p. 1333]: N is prime if there is D such that $(D/N) = -1$, i.e. if there is "genuine" complex structure on *Z/NZ* (not all residues are quadratic). The "propagation" of this test down the structure-tree of an integer seems automatic, yet intriguing; the converse of Fermat's Theorem [38] is investigated in terms of the direct relation between the factorization of N and the properties of the prime factors of *Aut(Z/NZ)* (i.e. $p|(N-1) \to (q|N)$). It is interesting to investigate how this converse behaves under PoSet descent for the "bundle" *Aut(Z/NZ)* acting on *Z/NZ*. For additional info on large primes of the form $2^k \cdot n + 1$, see [33].

5.1. Finite Abelian Groups as a Shadow of Permutation Groups

It seems that dealing with Z_n we are "safe" in the commutative realm ... Not quite! Z_n as an abelian group is the set $[n]$ with a cyclic order; then the permutation group S_n acts on it. Moreover the action of $Aut(Z_n) = U_n$ is by left multiplication with generators from U_n which induce a permutation which breaks the set into orbits.

In fact since 0 is is "uninteresting" (left fixed), this is a context for projective geometry. For the prime numbers, the projective space $U_n = Z_p^*$ is the union of such orbits.

Therefore we should study the inclusion

$$Aut(Z_n) \to S_{n-1}$$

and the implications on the geometry of the Klein space Z_n. Since "disjoint union" of spaces is not interesting, one focuses on p-groups and finite fields Z_p:

$$Aut(Z_p) = Z_p^* \to S_p$$

Consider Z_7 for example; the first level of structure $7-1 = 2 \cdot 3$ (projective geometry of $X = Z_7^*$) is as follows.

Multiplication by 2, i.e. $\phi_2(k) = 2k$ yields two orbits of 3 elements, reflecting the top level of structure in the hierarchy tree $7 = 2 \cdot 3 + 1$, or otherwise put

$$Aut(Z_7^* \equiv Z_6 \equiv Z_2 \times Z_3.$$

There are obvious implications of quadratic reciprocity ("complex structures" or higher "rotation"/ equivariant structures on Z_p).

5.2. Primes and Quarks

More importantly is the possible connection between the "hidden" hierarchy of structure of the primes and quarks, as noted in [7].

The equivariant theory involves covering spaces which in the larger context of permutation groups, leads to a full sequence of "quark-like" objects (besides the S_3-quarks). Indeed, the circle (quantum phase) is just a limit of Z_n. The "ultimate theory" *is* algebraic number theory (via categorification), so The Resonance Model / QC will revert to Quantum Fourier Transform and Dirichlet Transform (additive and multiplicative number theory); then besides Z_4 (complex numbers) and Z_6 (quark symmetry), all modes of vibration will be involved (although SU_3 color symmetry, not quark flavor, is still important, since it reflects the 2 +2* = 3 +1 correspondence).

The tree structure of primes is probably related to the pre-Lie algebra of rooted trees. The doubling at a node is in some sense the "classical information", while the +1 operation is the "quantum information", corresponding to linearization; the reverse operation, taking the projective space $Z_n^* = Aut(Z_n)$. Here the "lines" are in fact the Z-submodules (finite circles), with generators $Aut(Z_n)$, so that the projective space PZ_n seams to correspond to one level of hierarchy in the structure-tree of the prime.

If primes have multiplicity, then these edge labels of the tree lead to an interpretation of these trees as SP-graphs.

Recalling that entropy is the expectation value of information S = < I >, and that the quantity of information is $I = \log|Aut(System)|$, then the relation with the Primon Model is of interest (Energy levels $E_p = \log p$ and temperature $1/s$ etc.).

For instance the quantity of symmetries of a primon (information as a "charge", with its equivalent of energy, and thus "rest mass"):

$$E_p = I(p) = \log|Aut(Z_p)| = \log\phi(p) = \log(p-1).$$

Classical information $p = 2$ seems to be "special": no mass, clonable, etc.

Returning to Riemann zeta function and the Path Integral on SP-graphs, we note another point of interest. The higher Green functions (correlators) correspond to $\zeta(2k)$, and symmetric functions / trivial zeros of ζ. How is this related to the above rooted trees (or SP-graphs)?

It is expected that "quarks" are not grouped by generations, since flavor symmetries are "too broken" to be true in some sense, and rather correspond to such rooted trees (the primes, or their underlying structure of symmetries, as Klein spaces; projective spaces hierarchy). Then, only the (odd) primes of the form 2^n+1 (leaves of the rooted trees) should be regarded as "true quarks" (elementary). Since then $n = 2^k$, they "are" the Fermat primes (if the "trivial prime" 2 is included; $Aut(Z_2) = 0$):

$$u: F_{-\infty} = 2,\ d: F_0 = 3,\ s: F_1 = 5,\ c: F_2 = 17,\ b: F_3 = 257,\ t: F_4 = 65537.$$

Now compare them with the rest masses in MeV (Wikipedia):

2.4, 4.8, 104, 1, 270, 4, 200, 171, 200…

Recall that, as a tentative:

Conjecture 5.1. *There are only five Fermat primes, and therefore only six quark flavors.*

Then, instead of "generations", triggered by the closeness of masses of u and d quarks, and the misinterpretation of flavor as color, there are many *quark levels of structure*, corresponding to the internal symmetry structure of primes (basic finite fields).

5.3. Primes as PoSets and Riemann Hypothesis

Regarding the non-trivial zeros of the Riemann zeta function (see 4.7), the average density of zeros interpreted as eigenvalues E_n of a hermitian operator [34] ($s_n = ½ + iE_n$):

$$p(E) = \frac{1}{2\pi}\ln(E/2\pi)$$

is the information content $p(\sigma) = \ln\sigma$ (compare with Equation 1). This is expected in terms of the empirical evidence leading to the *Gaussian Unitary Ensemble Hypothesis*. (GUE) [35].

The zeros seem random since primes, which are graph-like objects, seem random when projected on the integer lattice.

The structure-trees of primes, when evaluated to the corresponding prime number, yield a PoSet structure on the set of prime numbers: $p- > q$ if the tree-structure t_q of q is a sub-tree of the tree-structure t_p of p. For example, if $p = 2^k \cdot q_1^{e_1} \ldots e_m^{e_m} + 1$ then $p \to q_i$.

With the five Fermat primes as the "atoms", the first level of primes includes 7, 11, 13, 19, while $23 = 1 + 2 \cdot 11$ is on the 2nd level $23 \to 11 \to 5$, together with $29 \to 7 \to 3$.

Since for example $31 \to 3$ and $31 \to 5$, a lattice structure arises (POSet), characteristic to a divisibility structure (Hopf structure).

For example, with the atomic elements pictured at the boundary of the POSet (the three of the five Fermat primes shown), the lower portion of the POSet of primes is:

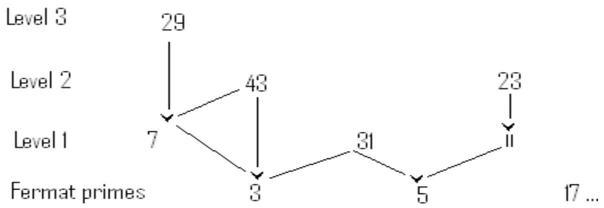

Fig. 3.

The power k of 2, representing the binary content ($p=2^k n + 1$), is encoded in the difference of "energy" levels, e.g. $29 = 1 + 2^2 \cdot 7$ sits at level of *7* plus two "stories" higher, while

$$\log \zeta(s) = -\sum_{p \in P} \log(1 - 1/p^s)$$

is one level higher then *3* and *5*.

The POSet contains unoriented cycles (not simply connected), also like the above triangle with vertices 43, 7, and 3, and it is capable of forming "proper modes"/ periods:

$$p \to H_p \to Spec(H_p), \quad E_n = h\omega_n, \quad length \sim frequency,$$

where $$length = \sum orbits = \sum e_k p_k$$

is a Lie "algebra" cochain (additive number theory; multiplicative partitions of n), e.g. Z_p^e has length $\log(p^k) = k\log(p)$. The relation between multiplication and a corresponding operation of trees suggests:

$$(p-1)(q-1) = pq - (p+q) + 1 \Rightarrow t_{p \cdot q} - t_p \cdot t_q = t_p + t_q,$$

in other words:

$$(\delta t)(p,q) = t_p + t_q.$$

Since this process reflects the orbit structure of symmetry groups$(Aut(Z_n))$, then it is a natural candidate of study as a *dynamical system*, with techniques of both thermodynamic partition functions and combinatorial [36, 37].

5.4. Dynamic Zeta Functions

The only point we would like to note (for now), is that the partition function

$$Z(\beta) = \sum_{states\,\sigma} e^{-\beta H(\sigma)}$$

is a state sum, while the generating sum is its logarithm [36, p. 3, 9]:

$$\exp(\sum Z_n / n) = \sum_{\sigma \in UEA(P)} t\sigma.$$

Then, we should apply Milnor-More Theorem for graded Lie algebras (and associated framework), to the special case of Dirichlet duality and transform (and zeta function).

As expected via Euler's formula, the analytic generating function is

$$\log \zeta(s) = -\sum_{p \in P} \log(1 - 1/p^s)$$

i.e. the prime zeta function.

6. Conclusions and Further Developments

We stop at this point the panorama of topics which bind physics, notably the discrete path integral approach to the fine structure constant, to number theory, notably the quantum duality of the integers and Riemann hypothesis, in order to recap a few of the main ideas.[8]

The Primon Gas model, as a number theory construct, is interpreted as describing a quantum system: the Quantum Computing model of the Hydrogen atom (QID; unified particle-field framework; Hopf bundle).

The corresponding discrete version Feynman Path Integral (generalization of convolution algebra to groupoids) provides a number theoretical foundation of the coupling constant (propagator). It allows relating the fundamental charges and fine structure constant to prime numbers (and Riemann zeta values), as quantum harmonic analysis (Dirichlet Transform).

A development of the Dirichlet duality for the universal Hopf ring **Z** would probably benefit from applying Tanaka-Krein duality to the corresponding braided category *C(Z)*. This would probably help in relation with Riemann hypothesis (poles of the Dirichlet Transform of Mobius function as the fermionic partition function).

That the primes have an internal structure when considering the Klein geometry of the corresponding projective spaces is to be evaluated in terms of the correspondent Integer CFT (via duality and Riemann Gas model).

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