Derivation of the Schrödinger Equation from the Laws of Classical Mechanics Taking into Account the Ether

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In the present work the author suggests a return to the idea of "hidden variables" as a physical field (ether). It is shown below that the Schrödinger equation can be derived from the deterministic laws of classical mechanics under the assumption that the ether exists. The reasoning is based to a great extent on the works of N.G.Chetaev. Formulas derived in the present work for the velocity of an elementary particle and for the forces exerted on the particle by the ether coincide precisely with those derived by V.A. Kotel'nikov using the standard probabilistic interpretation of the Schrödinger equation. Therefore, the proposed classical approach agrees with the probabilistic one, and hence with standard experiments. The mathematical development of the proposed model when applied to an atom leads to the idea that structures are formed in the ether inside the atom. From the standpoint of this model, De Broglie's "law of phase harmony" has a new physical interpretation.

1. Hidden Variables

As is well known, the Schrödinger equation describes many observations very well, but there are still heated discussions among scientists about its physical interpretation.

Currently, the probabilistic interpretation is the most common one. Its proponents, however, have difficulty explaining the results of experiments with nonclassical optical effects (e.g., the two-photon interference, teleportation of polarization of the photon, etc.) The probabilistic approach in these cases leads to negative probabilities. Note that any known interpretations of the quantum formalism for the case of "nonclassical" light are also inconsistent with the main concepts of special relativity.

The probabilistic interpretation is also unable to describe the behavior of quantum systems in living matter (biomolecules) [1]. E. Schrödinger was the first to write about this conundrum: "A single group of atoms existing only in one copy produces orderly events, marvelously tuned in with each other and with the environment according to most subtle laws... we are here obviously faced with events whose regular and lawful unfolding is guided by a 'mechanism' entirely different from the 'probability mechanism' of physics".

Historically, the issue of incompleteness in the description of physical reality by quantum mechanics was put forward for the first time by Einstein, Podolsky, and Rosen in 1935. They proposed the existence of «hidden variables», that is such properties of elementary particles that allow a quantum system's consistency with the deterministic theory of elementary particles. They also referred to the hidden variables as being "local". Later it was proven that «hidden variables» can be either: 1) "nonlocal" (the "nonlocally" is the existence of a connection between any spatially separated measurement devices) or 2) a field of a special type wherein disturbances can spread at speeds greater than the speed of light. In general both hypotheses are inconsistent with the main concepts of special relativity.

In order not to be in conflict with special relativity the majority of the physics community accepted the point of view that the probabilistic description of quantum mechanics cannot be avoided. Followers of the Copenhagen school (which is the most widely accepted) insist on the point of view that physics is the science which rests solely with measurements. Questions like "where was a particle before it was detected by a device" does not make sense from this viewpoint.

According to quantum mechanics, the particle's state, at any given time, is completely determined by its wave function. Two processes can effect a wave function if its evolution is in accordance with 1) the Schrödinger equation, and 2) the process of measurement. Disagreements concerning the interpretation of the measurement process arise even within the Copenhagen school. There are two views regarding the wave function: on the one hand it can be viewed as a physical aspect of reality that undergoes a "collapse" as a result of a measurement; on the other hand the wave function can be used as simply a mathematical tool whose sole purpose is its role in calculating the probability of experimental results. The above two viewpoints leave a number of questions open.

As a result more and more physicists are inclined towards neither interpretation, which was expressed by physicist David Mermin as "Shut up and calculate!"

One of the basic assertions of quantum theory is that as a result of any measurement process we can obtain only one of two complementary pairs precisely: either spatial position or momentum in one case, versus energy or time in another case, up to the precision allowed by the Heisenberg principal. Since any measurement disturbs the system both members of a complementary pair cannot be resolved, i.e. we cannot assign a particle a position and a velocity obtained from different measurements, even when the state of the particle is described by the same wave function. However, it would not conflict with the mathematical formalism of quantum mechanics and the results of experiments if one assumes that velocity and position of the particles exists *a priori* (before measurement).

The opinion of the author is also in concert with the idea that hidden variables exist. It is proven below that the Schrödinger equation can be derived solely from the deterministic laws of classical mechanics. N.G. Chetaev (1936) showed in his works [2] the analogy between the equation describing the stable motion of a mechanical system, under the action of conservative forces and Schrödinger's equation. In the author's view, this analogy is not coincidental. Further mathematical development of Chetaev's works, as shown below, leads to the conclusion that the motion of elementary particles can have a deterministic interpretation if the influence of a particular medium (ether) is taken into consideration.

An article by V.A. Katel'nikov on hidden variables was recently published posthumously in *Physics-Uspekhi* [3]. Katel'nikov based his work on the Schrödinger equation and the probabilistic interpretation of the ψ function. He derived the velocity and trajectory of a particle under the assumption that its motion satisfies the Schrödinger equation. He showed that in this case the particle should be moving under the influence of two forces: a classical force, described by the potential U, and a "quantum" force exerted on the particle by a "quasifield" (the term introduce by Katel'nikov). The formulas for the particle's velocity and the "quantum" force obtained by Katel'nikov, coincide completely with the formulas obtained in this work. The approach of classical mechanics is therefore shown to be in agreement with the probabilistic one, and hence with experiments.

2. The Schrödinger Equation as a Condition of Stability of a Mechanical System

The Schrödinger equation is a postulate of quantum mechanics. One can attempt to derive a postulate in order to look for a way to "extend the existing model", that is to bring new concepts into science, and to look at the problem by taking into account deeper processes of nature. The derivation, presented below, **does not require the introduction of new postulates in physics**, which makes it especially interesting. The Schrödinger equation is derived from the laws of deterministic classical mechanics, which allows an interpretation from a deterministic point of view rather than from the traditional probabilistic view. The reasoning in this chapter is based a great deal on the works of Chetaev [2].

Consider the motion of a mechanical system under the action of conserved forces that do not depend on time *t* explicitly. In this case the kinetic energy of the system can be represented as follows:

$$2\mathrm{T} = \sum_{i,j} a_{ij} p_i p_j \,, \tag{1}$$

with which is associated the energy integral

$$H = T + U = \varepsilon = \text{const} \,, \tag{2}$$

where $U(q_1,...,q_n)$ is potential energy , $q_1,...,q_n$ are generalized coordinates, and $p_1,...,p_n$ are generalized momenta.

As is known, Newton's equation for such a system using Hamilton's equations takes the form

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \tag{3}$$

Eqs. (3) are the equations for characteristics for the following partial derivative equation of the first order (known as the Jacobi equation):

$$\frac{\partial S}{\partial t} + H\left(q_1,...,q_n;\frac{\partial S}{\partial q_1},...,\frac{\partial S}{\partial q_n}\right) = 0 \tag{4}$$

If the energy integral exists and the force function does not depend explicitly on time, then the complete integral of the Jacobi equation takes the following simple form:

$$S = -\varepsilon t + V\left(q_1,...,q_n;a_1,...,a_n\right) \tag{5}$$

Equation (5) yields a system of canonical equations:

$$\begin{aligned} \frac{\partial S}{\partial a_i} &= \frac{\partial V}{\partial a_i} = b_i, \quad i = 1,2,...,n-1 \\ \frac{\partial S}{\partial q_i} &= \frac{\partial V}{\partial q_i} = p_i, \quad i = 1,2,...,n \end{aligned} \tag{6}$$

Let us introduce a function $\psi(S)$. In view of (5), the following equations result: $\psi_{tt} = \psi''\varepsilon^2$ (here $\psi' \equiv \frac{d\psi}{dS}$) , $\psi_t \equiv \frac{d\psi}{dt}$

$$\frac{\partial}{\partial q_i}\left(a_{ij}\frac{\partial \psi}{\partial q_j}\right) \equiv \frac{\partial}{\partial q_i}\left(\psi' a_{ij}\frac{\partial S}{\partial q_j}\right) = \psi'' a_{ij}\frac{\partial S}{\partial q_i}\frac{\partial S}{\partial q_j} + \psi'\frac{\partial}{\partial q_i}\left(a_{ij}\frac{\partial S}{\partial q_j}\right) \tag{7}$$

From this result it follows that

$$L\left[\psi\right] \equiv \sum \frac{\partial}{\partial q_i}\left(a_{ij}\frac{\partial \psi}{\partial q_j}\right) = \psi''\sum a_{ij}p_i p_j + \psi' L\left[S\right] \tag{8}$$

If we represent the energy integral (2) as

$$\sum a_{ij}p_i p_j = 2(\varepsilon - U)\,, \tag{9}$$

then equation (8) can be written as

$$L\left[\psi\right] = \psi_{tt}\frac{2\left(\varepsilon - U\right)}{\varepsilon^2} + \psi' L\left[S\right] \tag{10}$$

Now we explore the mechanical motion of the system described by Eqs. (3) for stability. As is known from the theory of stability, the variational Poincaré equation with respect to a variation of $q_1,...,q_n$ has the form:

$$\frac{d\xi_i}{dt} = \sum \frac{\partial}{\partial q_j}\left(a_{is}\frac{\partial S}{\partial q_s}\right)\xi_j \tag{11}$$

Assume now that under some particular set of initial conditions the non-perturbed object's motion is stable. Then the characteristic number for the expression

$$\exp\int \sum \frac{\partial}{\partial q_r}\left(a_{rj}\frac{\partial S}{\partial q_j}\right)dt = \exp\int L\left[S\right]dt \tag{12}$$

must be zero. In the primitive case this requirement is satisfied if

$$L[S] \equiv \sum_{i,j}\frac{\partial}{\partial q_i}\left(a_{ij}\frac{\partial S}{\partial q_j}\right) = 0 \tag{13}$$

Thus for stable motion (and only for it) Eq. (10) simplifies and takes the form:

$$\psi_{tt}\frac{2\left(\varepsilon - U\right)}{\varepsilon^2} = L\left[\psi\right] \tag{14}$$

Let us search for a solution of the above equation for the motion of a particle with mass *m*. Denote by $\vec{r}$ a position vector of a particle. From all possible functions ψ let us look for those that can be represented in the form:

$$\psi = \exp\left(\frac{-i}{\hbar}S\right) = \exp\left(\frac{-i\varepsilon}{\hbar}t\right)\psi(\vec{r}) \tag{15}$$

Substituting (15) into Eq. (14), the latter takes the form of the Schrödinger equation.

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi(\vec{r},t) + U(\vec{r})\psi(\vec{r},t) \tag{16}$$

We now consider a special case of conservative forces that do not depend on time t explicitly here. Applying substitution (15) once again to the last equation, we obtain the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\Delta\psi(\vec{r}) + U(\vec{r})\psi(\vec{r}) = \varepsilon\psi(\vec{r}) \tag{17}$$

This derivation of the Schrödinger equation encounters a problem when applied to the hydrogen atom. In the case of a potential $U \propto 1/r$ the solutions of Eq. (3) are elliptical trajectories. However elliptical trajectories that satisfy the conditions of stability (14) do not satisfy equation (17). Thus, we must search for such additional conservative forces with potential $W(q_1,\dots,q_n)$ that make it possible to write the condition of stability (14) in the form of the stationary Schrödinger equation (17) (i.e. which do not contain W explicitly). Note, that such forces are not small.

3. Forces Exerted on a Particle by Ether

Assume that in addition to the primary forces represented by potential $U(q_1,\dots,q_n)$ there are also forces characterized by a potential $W(q_1,\dots,q_n)$. Now consider the stability of such motion under action of the above two forces with respect to a variation of coordinates $q_1,\dots,q_n$.

We now represent the time-independent component of the function $\psi(S)$ as

$$\psi(\vec{r}) = A\exp(iV/\hbar), \tag{18}$$

where A is some real function of coordinates q_i. Combining all of the above yields

$$\frac{\partial V}{\partial q_j} = \frac{\hbar}{i}\left(\frac{1}{\psi}\frac{\partial\psi}{\partial q_j} - \frac{1}{A}\frac{\partial A}{\partial q_j}\right) \tag{19}$$

Consequently, in view of (6), the expression (13) takes the form

$$\sum_{i,j}\frac{\partial}{\partial q_i}\left[a_{ij}\left(\frac{1}{\psi}\frac{\partial\psi}{\partial q_j} - \frac{1}{A}\frac{\partial A}{\partial q_j}\right)\right] = 0 \tag{20}$$

Conservation of energy can thus be expressed through the function ψ as:

$$-\frac{\hbar^2}{2}\sum_{ij}a_{ij}\left(\frac{1}{\psi}\frac{\partial\psi}{\partial q_i} - \frac{1}{A}\frac{\partial A}{\partial q_i}\right)\left(\frac{1}{\psi}\frac{\partial\psi}{\partial q_j} - \frac{1}{A}\frac{\partial A}{\partial q_j}\right) = \varepsilon - U - W \tag{21}$$

Adding the above equation and Eq. (20), the necessary condition of stability in the first approximation takes the form

$$\begin{aligned}&\frac{1}{\psi}\sum_{i,j}\frac{\partial}{\partial q_i}\left(a_{ij}\frac{\partial\psi}{\partial q_j}\right) - \frac{1}{A}\sum_{i,j}\frac{\partial}{\partial q_i}\left(a_{ij}\frac{\partial A}{\partial q_j}\right) - \\ &-\frac{2}{A}\sum_{i,j}a_{ij}\frac{\partial A}{\partial q_j}\left(\frac{1}{\psi}\frac{\partial\psi}{\partial q_i} - \frac{1}{A}\frac{\partial A}{\partial q_i}\right) + \frac{2}{\hbar^2}(\varepsilon - U - W) = 0\end{aligned} \tag{22}$$

If function A satisfies

$$-\frac{1}{A}\sum_{i,j}\frac{\partial}{\partial q_i}\left(a_{ij}\frac{\partial A}{\partial q_j}\right) - \frac{2i}{\hbar A}\sum_{i,j}a_{ij}\frac{\partial A}{\partial q_j}p_i - \frac{2}{\hbar^2}W = 0, \tag{23}$$

then Eq. (22) does not contain w. The expression (23) can be split into real and imaginary parts:

$$W = -\frac{\hbar^2}{2A}\sum_{i,j}\frac{\partial}{\partial q_i}\left(a_{ij}\frac{\partial A}{\partial q_j}\right) \qquad \sum_{i,j}a_{ij}\frac{\partial A}{\partial q_j}p_i = 0. \tag{24}$$

Thus if the potential w has the structure defined by Eqs. (24), the condition of stability (22) takes the form of the stationary Schrödinger equation:

$$-\frac{\hbar^2}{2\psi}\sum_{i,j}\frac{\partial}{\partial q_i}\left(a_{ij}\frac{\partial\psi}{\partial q_j}\right) + U = \varepsilon \tag{25}$$

For a case of one particle Eq. (25) takes the form of Eq. (17).

Eq. (18) and Eq. (6) combined give the speed of the particle in the form

$$\vec{v}(\vec{r}) = \frac{1}{m}\nabla V \tag{26}$$

The additional force acting on the particle is equal to

$$\overline{F}_q = -\nabla W = \frac{\hbar^2}{2m}\nabla\frac{\Delta A}{A} \tag{27}$$

Combining Eq. (18), the condition of stability (20), Eq. (25) and Eq. (27), the time-independent Schrödinger equation (25) takes the form of the law of conservation of energy of stable orbits.

$$\frac{1}{2}mv^2 + W + U = \varepsilon. \tag{28}$$

Thus, if a particular solution of the Schrödinger equation is given, the velocity of the particle and the expression for the additional force that must act on the particle along stable trajectories can be determined from Eq. (26) and Eq. (27). Note that this force depends on the velocity and the form of the trajectory, which is typical for the motion of a body in a continuous medium.

Conclusion. Any description of the motion of an elementary particle using the laws of deterministic classical mechanics must include forces that are exerted on the particle on quantum orbits by a medium (ether).

4. Electron in the Field of an Atomic Nucleus

Using the above approach, let us analyze an electron in the Coulomb field in a hydrogen atom, taking the nucleus to be at rest. The time-independent Schrödinger equation has the form

$$-\frac{\hbar^2}{2m}\Delta\psi(\vec{r})+\left(\varepsilon+\frac{\alpha}{r}\right)\psi(\vec{r})=o \tag{29}$$

where $\alpha=e^2/4\pi\varepsilon_0$, and e is the charge of the electron. Let us show that the well known solution to Eq. (29) can be obtained purely by mathematics without relying on any of the postulates of conventional quantum mechanics.

Recall that the Laplace operator in spherical coordinates has the form

$$\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} \tag{30}$$

We look for a solution of Eq. (29) of the form

$$\psi\ (r,\vartheta,\varphi)=R\ (r)\cdot Y\ (\vartheta,\varphi)\ .$$

First note that Eq. (29) can be rewritten as:

$$\begin{aligned}&\frac{\hbar^2}{2mR}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+r^2\left(\varepsilon+\frac{\alpha}{r}\right)=\\&\frac{\hbar^2}{2mY}\cdot\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y}{\partial\vartheta}\right)+\frac{\hbar^2}{\sin^2\vartheta}\frac{\partial^2 Y}{\partial\varphi^2}\end{aligned} \tag{31}$$

Note also that the variables are now separate: the left hand side of the equation contains only r and the right hand side contains only ϑ and φ. Eq (31) must hold for any r, ϑ and φ. This is possible only if the left and right sides of the equation are both equal to the same constant C. Therefore, Eq. (31) can be replaced by the following two equations:

$$\frac{\hbar^2}{2m}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+r^2\left(\varepsilon+\frac{\alpha}{r}\right)R=CR \tag{32}$$

$$\frac{\hbar^2}{2m}\cdot\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y}{\partial\vartheta}\right)+\frac{\hbar^2}{2m\sin^2\vartheta}\frac{\partial^2 Y}{\partial\varphi^2}=CY \tag{33}$$

First, we will look for finite and single-valued solutions of the equation (33). The condition of being single-valued can be expressed as

$$Y(\vartheta,\varphi+2\pi)=Y(\vartheta,\varphi) \tag{34}$$

Again we solve equation (33) by separation of variables. Let $Y(\vartheta,\varphi)=\Theta(\vartheta)\cdot\Phi(\varphi)$. Then (33) can be written as

$$\frac{1}{\Theta}\sin\vartheta\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Theta}{\partial\vartheta}\right)+\frac{2mC}{\hbar^2}\sin^2\vartheta=-\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=k^2 \tag{35}$$

where k is some constant. The last equation is decomposes into two:

$$\sin\vartheta\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Theta}{\partial\vartheta}\right)+\frac{2mC}{\hbar^2}\cdot\Theta\cdot\sin^2\vartheta=k^2\cdot\Theta \tag{36}$$

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=k^2$$

The second equation above yields the immediate solution: $\Phi_k=C_1\cdot\exp(ik\varphi)$. To satisfy the condition of the solution's being single-valued (35), k must take the integer values $k=0,\pm1,\pm2,...$ In the equation (36) let us make the substitution $x=\cos\vartheta$. After the substitution, denote function Θ by y. After the changes, (36) takes the form:

$$\frac{d}{dx}\left[(1-x^2)\cdot\frac{dy}{dx}\right]+\left(\frac{2mC}{\hbar^2}-\frac{k^2}{1-x^2}\right)y=0 \tag{37}$$

The theory of differential equations shows that the only solutions of (37) that are finite in the interval [-1,+1] are the adjoint Legendre polynomials

$$P_l^k(x)=(1-x^2)^{\frac{|k|}{2}}\cdot\frac{d^{|k|}}{dx^{|k|}}P_l(x)\,,$$

$$P_l(x)=\frac{1}{2^l\cdot l!}\cdot\frac{d^l}{dx^l}\left[(x^2-1)^l\right]\quad l=0,1,2,3,... \tag{38}$$

Since the $(l+1)^{th}$ derivative of a polynomial of degree l is zero, the functions (38) are non zero only for $|k|\le l$. These polbynomials are eigenfunctions of the differential equation (37) and

$$\frac{2mC}{\hbar^2}=l(l+1) \tag{39}$$

are its eigenvalues. Thus from this very brief derivation we see that the quantization of angular momentum is obtained strictly mathematically and does not require the introduction of any additional axioms (other than finiteness of the solution). **It is important to emphasize here that Planck's reduced constant $\hbar$ appears in all solutions only due to substitution (15).**

Substituting (39) into the Eq. (32) yields

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2m}{\hbar^2}r^2\left(\varepsilon+\frac{\alpha}{r}\right)R=l(l+1)R \tag{40}$$

The expressions $R_{n,l}(r)$ for the radial part $\psi(r,\vartheta,\varphi)$ can be obtained by taking into account the fact that $R\ (r)\cdot$ is bounded at infinity. The expressions for the eigenvalues turn out to be the well-known formulas obtained by Bohr for a hydrogen atom.

$$\varepsilon=-\frac{\alpha^2 m}{2\hbar^2 n^2}\qquad n=1,2,3,... \tag{41}$$

Thus, in the case of a hydrogen atom Schrödinger function has the form:

$$\psi_{nlk}\ (r,\vartheta,\varphi)=\exp(-i\frac{\varepsilon_n}{\hbar}t)R_{n,l}\ (r)\Theta_{l.k}(\vartheta)\exp(ik\varphi) \tag{42}$$

Given (18) and (42), the phase of the wave function takes the form $\beta=-\frac{\varepsilon_n}{\hbar}t+V/\hbar=-\frac{\varepsilon_n}{\hbar}t+k\varphi$.

Given the above expression and Eq. (26) the velocity vector of the electron equals to

$$\vec{v}(\vec{r})=\frac{\hbar k}{m}\nabla\varphi\quad\text{or}\quad\vec{v}=\frac{\hbar}{m}\frac{k}{r\sin\vartheta} \tag{43}$$

Thus, the electron moves along a circle lying in the xy plane with its center on the z-axis. Using the notation $r\sin\vartheta=\rho$, $\vec{v}$ takes the form $\vec{v}=\frac{\hbar}{m}\frac{k}{\rho}$. This expression is similar to the expression for the velocity of particles in a case of the potential circular motion of fluid with circulation $\Gamma = \frac{\hbar k}{m}$. The net force that provides centripetal acceleration equals $\frac{\hbar^2 k^2}{m\rho^3}$.

Let us consider the solution of the Schrödinger equation when the speed of a particle is zero, for example $\psi_{100} = \text{const} \times \exp(-r/r_0)$. In this case the Coulomb force of attraction is balanced by a force of repulsion exerted on the particle by a structure formed in ether.

5. V.Kotel'nikov's Model of Nonrelativistic Quantum Mechanics

Naturally, a question arises: how well does this interpretation of the Schrödinger equation agree with the probabilistic interpretation?

Recently a work by the well known Russian physicist V.A. Kotel'nikov was published [2], which is also devoted to the search for hidden variables. In his study V.A. Kotel'nikov as a starting point uses the Shrödinger equation and probabilistic interpretation of the ψ function. V.A. Kotel'nikov raised a question: how can an elementary particle move according to the laws of classical mechanics if its probabilistic behavior is determined by the Schrödinger equation. As result of his mathematical derivations, he concluded that the particle should move under the action of two forces: a classical force, defined by the potential U and a " quantum" force as well. He suggested that a field, which he named the "quasifield", exists that accompanies the particle and produces the "quantum" forces. From Katel'nikov's standpoint, the "quasifield" is some type of a scalar field which can be modeled as gas or compressible liquid. It is important to note, that the formulas Katel'nikov obtained for the additional "quantum" force and for the particle's velocity coincides completely with the corresponding formulas (26) and (29) derived above. This means that our reasoning is in agreement with probabilistic results and the interpretation of ψ function, and therefore also with experiments.

Below we present the derivations of these formulas done by Katel'nikov. A small number of copies of his work, not completed because of his death, were published in Russia. Three chapters were published separately in the English version [3] of the Russion journal *Physics-Uspekhi (Advances in Physical Sciences).*

As is known, the basic assertion of nonrelativistic quantum mechanics is that the state of a particle at a given moment in time t is fully described by a wave function

$$\psi(\vec{r}) = A\exp(i\beta) \tag{44}$$

where $A(\vec{r},t)$ and $\beta(\vec{r},t)$ are real. The function $A(\vec{r},t)$ determines the probability that the particle at some instant of time t resides within a small volume dq, i.e.

$$dP = A^2(\vec{r},t)dq \tag{45}$$

and $\beta(\vec{r},t)$ determines the dynamic state of the particle.

Knowing ψ at the initial moment of time, and the potential energy associated with the external fields $U(\vec{r},t)$, one can find $\psi(\vec{r},t)$

for other moments in time using the Schrödinger equation.

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi(\vec{r},t) + U(\vec{r})\psi(\vec{r},t) \tag{16*}$$

Let us try to construct a model that corresponds to the above basic statement of quantum mechanics and hence to the experimental evidence but which also imply a certain trajectory of the particle, as is the case in macroscopic mechanics.

Suppose that at some moment of time t the particle is at a point with position vector $\vec{r}$ and has velocity $\vec{v}(\vec{r},t)$. Let us find the probability that during a time interval $(t,\ t+dt)$ the particle will cross a small area $d\vec{s}$. During the time interval dt, the particle moves by $\vec{v}(\vec{r},t)dt$. It will cross the area $d\vec{s}$ if at some instant of time t it was at a distance $\lambda\vec{v}(\vec{r},t)dt$ $(0<\lambda<1)$ from one of the points in this area or, in other words, if at time t it was within a volume $dq = \vec{v}\cdot d\vec{s}\,dt$ adjacent to the area $d\vec{s}$. According to formula (44), the probability of this event is $dP_{ds} = A^2\,\vec{v}\cdot d\vec{s}\,dt$. For $\vec{v}\cdot d\vec{s}<0$, the particle will cross the area $d\vec{s}$ in the opposite direction.

Let us choose some volume q bounded by a closed surface S. The probability that the particle will escape from this volume q, i.e., will cross the surface S within the time interval $(t,\ t+dt)$ is, according to the Ostrogradsky-Gauss theorem, given by

$$P_- = dt\int_S A^2\vec{v}(\vec{r},t)d\vec{s} = dt\int_q \nabla\cdot(A^2\vec{v})dq \tag{46}$$

The probability that at the moment of time t the particle resides within the volume q is, according to (45), equal to

$$P_t = \int_q A^2(\vec{r},t)dq$$

The probability that the particle will stay within volume q at time $t+dt$ can be expressed as

$$P_{t+dt} = \int_q (A + \frac{\partial A}{\partial t}dt)^2 dq = \int_q (A^2 + 2A\frac{\partial A}{\partial t}dt)dq \tag{47}$$

Here, we omitted the term with $(dt)^2$ as an infinitesimal of higher order of magnitude.

Evidently, the event "the particle is within volume q at some instant of time t " will be necessarily followed by either the event "the particle stays within volume q at some instant of time $t+dt$ " or the event "the particle leaves domain q within the time interval $(t,\ t+dt)$ ". Therefore, one finds that

$$P_t = P_{t+dt} + P_- \quad \text{or} \quad P_{t+dt} - P_t = -P_-$$

From this equality it follows that

$$\int_q \frac{\partial A^2}{\partial t}dq = -\int_q \nabla\cdot(A^2\vec{v})dq \tag{48}$$

and, since this equality should be valid for any q, one has

$$\frac{\partial A^2}{\partial t} = -\nabla(A^2 \vec{v}) \tag{49}$$

Now, let us find the value of $\partial A^2/\partial t$ according to the Schrödinger equation . We substitute $\psi(\vec{r},t)$ from formula (44) into Eq. (16*). Then we obtain the result

$$i\hbar\left(\frac{\partial A}{\partial t} + iA\frac{\partial \beta}{\partial t}\right) = -\frac{\hbar^2}{2m}\left[\Delta A + 2i(\nabla A)(\nabla\beta) + iA\Delta\beta - A(\nabla\beta)^2\right]\exp(i\beta) + Ua\exp(i\beta) \tag{50}$$

Cancelling both sides of the above equation by $\hbar\exp(i\beta)$ and setting the imaginary parts equal, we find that

$$\frac{\partial A}{\partial t} = -\frac{\hbar^2}{2m}\left[2A(\nabla A)(\nabla\beta) + A\Delta\beta\right] \tag{51}$$

Furthermore, multiplying both sides by $2A$, after some algebraic transformations we obtain

$$2A\frac{\partial A}{\partial t} = \frac{\partial A^2}{\partial t} = -\frac{\hbar^2}{2m}\left[4A(\nabla A)(\nabla\beta) + 2A^2\Delta\beta\right] \tag{52}$$

or $\frac{\partial A^2}{\partial t} = -\frac{\hbar}{m}\nabla(A^2\nabla\beta)$. The latter equation coincides with Eq. (49) if we assume that

$$v(\vec{r},t) = \frac{\hbar}{m}\nabla\beta(\vec{r},t) \tag{53}$$

Hence, if at some moment of time *t* a particle is at a point with position vector $\vec{r}$, its velocity should correspond to equation (53) in order that the Schrodinger equation and relation (45) be satisfied.

Let us now find the forces that should act on the particle to provide these velocities. Note that setting the real parts in Eq. (50) equal and canceling $A\exp(i\beta)$, we obtain

$$-\hbar\frac{\partial\beta}{\partial t} = \frac{\hbar^2}{2m}\left[-\frac{\Delta A}{A} + (\nabla\beta)^2\right] + U \tag{54}$$

Let us find the acceleration of the particle from the velocities (53). If a particle moves along a certain trajectory, so that $\vec{v}(\vec{r},t)$ depends on $\vec{r}$ and *t*, its acceleration and velocity are known to be related by the equation

$$\frac{d\vec{v}(\vec{r},t)}{dt} = \frac{1}{2}\nabla(v^2) - \vec{v}\times(\nabla\times\vec{v}) + \frac{\partial\vec{v}}{\partial t} \tag{55}$$

(as the total derivative of velocity is calculated in hydrodynamics)

According to Eq.(53), the particle velocity is a gradient of some function; therefore, the cross product $\nabla\times\vec{v}$ equals to zero and hence, one has

$$\frac{d\vec{v}(\vec{r},t)}{dt} = \frac{1}{2}\nabla(v^2) + \frac{\partial\vec{v}}{\partial t} \tag{56}$$

If we assume that the particle motion satisfies Newton's law, then the existence of acceleration requires a force acting on the particle:

$$\vec{F} = m\frac{d\vec{v}(\vec{r},t)}{dt} = \frac{m}{2}\nabla(v^2) + m\frac{\partial\vec{v}}{\partial t} \tag{57}$$

or, taking into account formula (53), we obtain

$$\vec{F} = \frac{m}{2}\nabla\left(\frac{\hbar^2}{m^2}(\nabla\beta)^2\right) + \hbar\frac{\partial\nabla\beta}{\partial t} = \frac{\hbar^2}{2m}\nabla(\nabla\beta)^2 + \hbar\frac{\partial}{\partial t}\nabla\beta \tag{58}$$

The expression on the right-hand side of Eq. (58) can also be obtained from relation (54). Indeed, calculating the gradients of the left-hand and right-hand parts of Eq. (54), we arrive at

$$\frac{\hbar^2}{2m}\nabla(\nabla\beta)^2 + \hbar\frac{\partial\nabla\beta}{\partial t} = \frac{\hbar^2}{2m}\nabla\frac{\Delta A}{A} - \nabla U \tag{59}$$

Taking this into account, we can rewrite expression for the force (58) as

$$\vec{F} = \frac{\hbar^2}{2m}\nabla\frac{\Delta A}{A} - \nabla U = \vec{F}_q - \nabla U = m\frac{d\vec{v}(\vec{r},t)}{dt} \tag{60}$$

Here, $\vec{F}_q$ is an additional force that should act on the particle to provide its motion according to the Schrödinger equation and hence there is an agreement with the experimental results. This force is a conservative force and is determined by the modulus of the wave function $A(\vec{r},t)$.

As one can see the formulas for the particle's velocity (53) and for $\vec{F}_q$ Eq. (60) coincides completely with Eqs. (26) and (27) that was obtained above using the approach of the classical mechanics. Note, that these equations are also valid in the case of the time-dependent Schrödinger equation

6. De Broglie's "Law of Phase Harmony" and its Physical Interpretation

Now we go back to the complete integral of the Jacobi equation (5). From the theory of differential equations, the geometric representation of the complete integral of a differential equation is a family of integral surfaces. Consider one integral surface. Through each point of this surface there is a unique characteristic line which is contained within the surface. The projection of this line onto a hyperplane describes a particular trajectory of a system. By giving various transformations $\psi = \psi(S)$, we obtain different functions ψ bound to the given trajectory. In this way the transition from Hamilton's equations to the Jacobi equation provides additional ways for describing the system behavior. If there exists some physical characteristic of the system, which from on the one hand side changes continuously along each trajectory, and on the other hand can be uniquely and continuously extended to the surrounding space, then the attempt can be made to describe such a variable characteristic mathematically by finding an appropriate $\psi = \psi(S)$.Note once again that substitution (15) is the one that allowed to extract from all possible trajectories those that satisfy Schrödinger equation, moreover it is due to this substitution that all solutions of the Schrödinger equation contain $\hbar$. The latter substitution was found empirically by Schrödinger as the one which yields the Rydberg-Ritz formula for the spectrum of the hydrogen atom.

The substitution (15) allows one to associate a function $\exp(-i\varepsilon t/\hbar)\psi(\bar{x})$ with each of the particle's trajectories. As it follows from the theory of differential equations for any fixed moment of time $t = t_0$ the surface.

$$\psi(t_0,\bar{x}) = \exp\left(\frac{-i\varepsilon t_0}{\hbar}\right)\psi(\bar{x}) = \text{const} \tag{61}$$

is perpendicular to the trajectories. This mathematical fact is the basis for the de Broglie-Bohm pilot-wave theory. De Broglie assumed that every elementary particle possesses an internal oscillatory process, although his theory does not address the nature of this process. He suggested that an oscillation with exactly the same frequency is also induced at each point of space surrounding the particle. Thus de Broglie introduces two objects: a particle and an accompanying stationary wave. Considering these objects separately and taking (61) into consideration he came to his famous "law of phase harmony": the phase of the ψ function associated with the moving particle must always be in accord with the phase of the stationary wave at the location of the particle.

The "stationary wave", however, requires a medium, presumably the "physical vacuum", in which to propagate. But the "vacuum" having non uniform energy density and pressure does not agree with the theory of relativity. De Broglie and his followers attempted to combine the pilot-wave theory and special relative, but they did not succeed.

As stated above the main goal of this paper is the search for "hidden variables". It was previously shown, that the Schrödinger equation can be interpreted as a condition of stability of a mechanical system if the ether is taken into consideration. In this case, structures can form in the ether which stabilize the motion of a particle in external fields. An example of a structure in simple liquids is a vortex. In superfluid He-3 structures like homogeneous precessing domain are observed, where the spins of the fluid particles precess with the same frequency and phase.

With the assumption that the ether has properties like a superfluid, a new interpretation of the "law of phase harmony" can be given: a term $\varepsilon / \hbar$ in the expression of the non-stationary component of the wave function phase represents on the one hand the precession frequency of spin of the electron, and on the other hand is a frequency of coherent precession of ether particles'. (Note that torque and energy have the same units of measure in physics.) This explains why the natural atomic frequencies change if a spin of electron is subjected to an external magnetic field.

The role of physics is to establish correspondence between mathematical properties of function ψ and physical quantities, i.e. to give it proper physical interpretation. At present the probabilistic interpretation of function ψ is the most common one. However, the problems of the probabilistic interpretation discussed in part 1, proposed "hidden variables" as an alternative interpretation of the Schrödinger equation.

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