A Detailed 'Wesley Evaluation' of the Pappas-Moyssides Experiments on Ampere's Bridge Compared to Jonson's Evaluation Using Coulomb's Law. Arguments Pro et Contra.

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Abstract
In this paper it is described how the Ampere Force Law can be applied to Ampere´s Bridge.
A detailed derivation based upon a paper by Wesley is being done.
The results are questioned by Jonson, who promotes a usage of Coulomb's law.
However, both proposals can account for forces of the order measured by Pappas and Moyssides in the early 1980s. It is up to the reader to choose model.
Finally, the law usually being used in order to predict forces between electric currents, the so-called Lorentz force, fails completely to predict the properties of the force.
This paper is mainly based upon a 20 years old paper by the author of this paper.

1. Motives for analysing electric circuits again.

It has widely been believed that basic electricity and magnetism was dealt with and finished until the end of the 19th century, though completed using the Special relativity Theory in the beginning of the 20th.

It is also widely believed that all the ‘great masters’ of electricity, as Coulomb, Ampère, Maxwell, Lorentz, Einstein et al all agree about basic matters. The new generations only add some features though defending principally the old theories.

However, it can easily be remarked that Ampère very early is presenting experimental results which are deeply inconsistent with the theories of Maxwell and Lorentz.

For example, Ampère describes [12], [23] how a metallic ‘boat’ is floating along the current in a mercury trough The Lorentz force, based upon Biot-savart’s law does not explain that movement. In this paper further evidence is given that the Lorentz force is unable to give credit to ‘parallel forces’ between currents.

Wesley principally succeeds in using Ampère’s law in this respect, whereas Jonson does the same using Coulomb’s law.

Hence, it seems to be an important actual task to begin analysing electric circuits and the laws explaining their behaviour. Back to the 19th century again!

2. Methodology

In order to attain an expression for the force between the two parts of Ampère's Bridge it is necessary to divide the work into several steps. Since the shape of the circuit is rectangular it is convenient to integrate the contribution to the total force between each linear part of the first part and the second part. This work has already been done in preceding papers [1], [6], [11] but the steps have not been accounted for individually. Due to the tedious work needed in order to replicate all the integrals, the detailed computations will hereby be presented, including all the steps. It is more fair play between scientists if making it easy for the reader to judge the claims with the aid of easily verifiable expressions.

Below both the theory given by Wesley [1], based upon Ampère's Law [12] and the theory given by Jonson [6], based upon Coulomb's Law, will be used in order to attain an expression for the force between the two parts of Ampère's Bridge.

2.1. Wesley's method.

Wesley [1] basically uses Ampère's law in his analysis of a set of Ampère'bridge..

It may also be mentioned that Jonson has been consulting Wesley himself in order to check that the integrals were performed in accordance with the method used by Wesley Regrettably, Wesley is not more among us. However, the paper by Wesley appeared to be very usable in order to define the integrals to be done.

2.1.1 Ampère's Bridge according to Wesley. Configuration being analysed.

The numbering of the branches of the bridge obeys the paper by Wesley [1], [20].

If going counterclockwise, the order by Wesley is 1,2,5,6,7,10 whereas Jonson uses the order 1,2,3,4,5,6 [9].

2.2. Jonson's method

Jonson [6] basically performs the integrals using the same mathematical definitions as Wesley. This is possible, since the integrations to be performed are straightforward integrals using Cartesian coordinates. Parts of the Wesley results can also be used in the integration work, since Jonson's theory uses the same term as one of Wesley's. This also indicates that there are many common features in the results of Wesley and Jonson. Otherwise it is unlikely that both the theories would have been able to predict measurement results with a reasonable accuracy.

3. The Check of the Wesley result for the force within Ampére's Bridge.

3.1. Definition of variables that appear in the expressions for the force upon Ampère's Bridge.

Since the approach by Wesley […] has appeared to be very usable in order to compute the contributions to the total Ampère force from each element of the circuit, it seems convenient to use the definitions of variables he gives in his article. But in the further analysis it has appeared necessary to add other completing, mainly geometric (Cartesian) variables.
Wesley gives two fundamental expressions for the computation of the force, one using line integrals, the other using volume integrals. The choice of form depends of whether the distances between the two parts of the circuit are close to each other or distant. In the first case the line approximation is inappropriate; in the other it is usable.
The expressions, referred to as 'Ampere's original differential force law' [1b] are as follows:

$$\frac{d^6\vec{F}}{d^3r_2d^3r_1}=\vec{r}(-2\frac{\vec{J}_2\bullet\vec{J}_1}{r^3}+3\frac{(\vec{J}_2\bullet\vec{r})(\vec{J}_1\bullet\vec{r})}{r^5}) \quad (1)$$

$$d^2\vec{F}=I_2I_1\vec{r}(-2\frac{(d\vec{s}_2\bullet d\vec{s}_{1)}}{r^3}+3\frac{(d\vec{s}_2\bullet\vec{r})(d\vec{s}_1\bullet\vec{r})}{r^5}) \quad (2)$$

It has to be remarked that Wesley prefers to use the coupling constant one instead of $\frac{\mu_0}{4\pi}$ which is customary. That causes some work at the very end when comparing the theoretical resultsdue to the formula and the measurement results, but that works, too, of course.
The first term will hereafter be referred to as the 'a term' and the second term the 'b term'.
To be noted is also that since the same current goes through the whole circuit (excluding the current source), $I_1=I_2$ and $|\vec{J}_1|=|\vec{J}_2|$, simpler written $J_1=J_2$ (3)
Further on, since the whole circuit is supposed to lie in one and the same geometrical plane, coordinates may be chosen so that z=0 (4)
Actual coordinates of parts of bridge being analysed will directly be picked from the figure in respective case without further reasoning, as is for example being done below in section 3.1.1 when inserting M as y variable.
Since in all cases the y component of the force is being analysed, for simplicity all the results due to Eq. (1) and (2) treated below will mean the y component.
Thus, instead of always writing $d^2\vec{F}\bullet\vec{u}_y=........$ it will simpler be written $d^2\vec{F}=........$

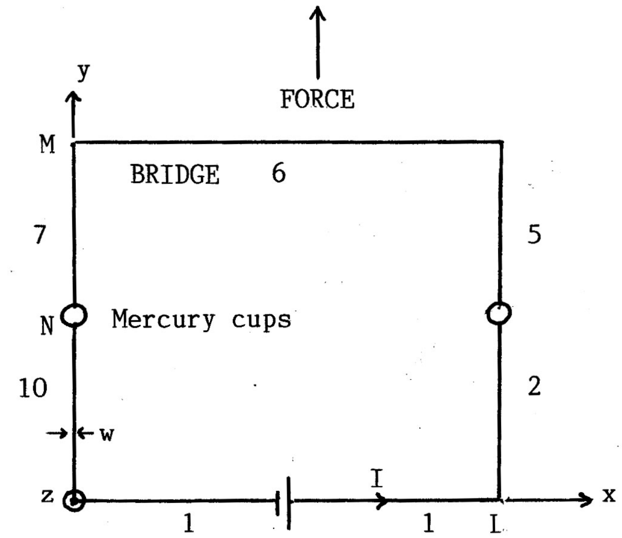

Figure 1: Ampère's Bridge [1], [20] (Wesley) , [9] (Jonson)

As can be inferred from the figure, the integrals involving the branches 2-5 and 10-7 respectively demand usage of Eq. (1) while all the other combinations demand Eq. (2). Please observe that the current source in the middle of branch 1 is denoted by the abbreviation 'CS'.
In previous papers [6], [11] it has just been referred to the integrals that they have been performed. So did also Wesley [1]. In the following sections these will be demonstrated.

3.1.1. Integral from branch 1 to 6(excluding the effect due to the current source in branch 1 see chapter 5.3.1))

In this case

$$d\vec{s}_1 = (dx_1, 0, 0) \qquad (5)$$

$$d\vec{s}_2 = (-dx_2, 0, 0) \qquad (6)$$

$$\vec{r} = (x_2 - x_1, M, 0) \qquad (7)$$

$$r = \sqrt{(x_2 - x_1)^2 + M^2} \qquad (8)$$

Performing the integration x_1 goes from 0 to L and x_2 too, goes from 0 to L.

3.1.1.1. The Second Ampere Law Term from Branch 1 to 6

Applying integral (2b)

gives $$d^2\vec{F}_{1\to 6,b} = I^2 M \int_{x_1}\int_{x_2} \frac{3(-(x_2 - x_1)^2)dx_1 dx_2}{((x_2 - x_1)^2 + M^2)^{\frac{5}{2}}} \qquad (9)$$

In order to solve the integration, the formula

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad (10) \qquad [13]$$

has to be applied, preferably by first integrating one step and rearranging the terms:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx} \qquad (11)$$

Defining $f = ((x_2 - x_1)^2 + M^2)$ (12)

Having a term $f^{-\frac{5}{2}}$, it seems reasonable to search for a primitive function $f^{-\frac{3}{2}}$, which after differentiation with respect to x_1 (the order of integration arbitrarily chosen) gives

$$-\frac{3}{2} f^{-\frac{5}{2}} \frac{df}{dx_1} \qquad (13)$$

Since $\frac{df}{dx_1} = -2(x_2 - x_1)$ (14)

expression (13) develops to $\frac{3(x_2 - x_1)}{((x_2 - x_1)^2 + M^2)^{2\frac{5}{}}}$ (15)

which appears to be equivalent to a dominant factor of the integrand.
The remaining factor is $-(x_2 - x_1)$ (16)

Letting the term $\frac{-3(x_2 - x_1)}{((x_2 - x_1)^2 + M^2)^{2\frac{5}{}}}$ be equivalent to the variable $\frac{dv}{dx}$ in the formula (10) above, $u = (x_2 - x_1)$ (17)

The integral (9) may now be identified with respective parts of Equation (11).
Hence,

$$d^2\vec{F}_{1\to 6,b} = I^2 M \int_{x_1=0}^{L} \int_{x_2=0}^{L} \frac{3(-(x_2 - x_1)^2) dx_1 dx_2}{((x_2 - x_1)^2 + M^2)^{\frac{5}{2}}} =$$

$$d^2\vec{F}_{1\to 6,b} = I^2 M \int_{x_1=0}^{L} \int_{x_2=0}^{L} \frac{3(-(x_2 - x_1))}{((x_2 - x_1)^2 + M^2)^{\frac{5}{2}}} (x_2 - x_1) dx_1 dx_2 \qquad (18)$$

thus equalling

$$I^2 M \Big(\int_{x_2=0}^{L} -((x_2 - x_1) dx_2 / ((x_2 - x_1)^2 + M^2)^{3/2})_{x_1=0}^{L} \Big) \quad -$$

$$\int_{x_1=0}^{L} \int_{x_2=0}^{L} \frac{dx_1 dx_2}{((x_2 - x_1)^2 + M^2)^{\frac{3}{2}}} \Big) \qquad (19)$$

The two right hand terms must be solved separately.
The second integral may be treated first due to its more simple form. This is preferably done by making the variable substitution $x_2 - x_1 = M \tan\varphi$ (20), which in turn implies that

$$-dx_1 = \frac{M d\varphi}{\cos^2 \varphi} \qquad (21)$$

The boarders are transformed as $x_1 = 0 \Rightarrow M\tan\varphi = x_2, \varphi = \tan^{-1}\frac{x_2}{M}$ and

$$x_1 = L \Rightarrow M\tan\varphi = x_2 - L \Rightarrow \varphi = \tan^{-1}\frac{x_2 - L}{M} \text{ respectively} \qquad (22)$$

Integrating first the right term with respect to x_1 gives

$$-I^2 M\int_{x_2=0}^{L} dx_2 \int_{\varphi=\tan^{-1}\frac{x_2}{M}}^{\tan^{-1}\frac{x_2-L}{M}} \frac{-Md\varphi}{\cos^2\varphi}\frac{1}{(M^2\tan^2\varphi + M^2)^{\frac{3}{2}}} = -I^2 M\int_{x_2=0}^{L} dx_2 \int_{\varphi=\tan^{-1}\frac{x_2}{M}}^{\tan^{-1}\frac{x_2-L}{M}} \frac{-\cos\varphi}{M^2}d\varphi$$

(23)

Realizing that $\sin\tan^{-1}\frac{x_2}{M} = \frac{x_2}{\sqrt{(x_2{}^2 + M^2)}}$ (24)

and $\sin\tan^{-1}\frac{x_2 - L}{M} = \frac{x_2 - L}{\sqrt{(x_2 - L)^2 + M^2)}}$ (25)

the expression (23) above simplifies into

$$I^2 M\int_{x_2=0}^{L} dx_2 \left(\frac{1}{M^2}\left(-\frac{x_2 - L}{\sqrt{(x_2 - L)^2 + M^2}} + \frac{x_2}{\sqrt{x_2{}^2 + M^2}}\right)\right) \quad (26)$$

This expression can easily be integrated, since the numerator is equal to the inner differential of the denominator with respect to the x_2 dependent term in both cases. Hence, (26)

simplifies to $-I^2 M\frac{1}{M^2}\left(\left(\sqrt{x_2{}^2 + M^2} - \sqrt{(x_2 - L)^2 + M^2}\right)\right)_{x_2=0}^{L}$ (27)

finally leading to $\frac{I^2}{M}(-2\sqrt{L^2 + M^2} + 2M)$ (28)

The first integral will now be treated:

$$I^2 M\left(\int_{x_2=0}^{L} -((x_2 - x_1)dx_2/((x_2 - x_1)^2 + M^2)^{3/2}\right)_{x_1=0}^{L} =$$

$$I^2 M\int_{x_2=0}^{L}\left(\frac{-(x_2 - L)}{((x_2 - L)^2 + M^2)^{\frac{3}{2}}} + \frac{x_2}{(x_2{}^2 + M^2)^{\frac{3}{2}}}\right)dx_2 \qquad (29)$$

The primitive functions are easily attainable and, hence, (29) develops to

$$I^2 M\left(1/\sqrt{(x_2 - L)^2 + M^2} - 1/\sqrt{x_2{}^2 + M^2}\right)_{x_2=0}^{L} \qquad (30)$$

which develops to $I^2 M\left(\frac{2}{M} - \frac{2}{\sqrt{L^2 + M^2}}\right)$ (31)

Now , by adding the first integral to the second, Eq.(18) can be finally simplified to

$$d^2\vec{F}_{1\to 6,b} = I^2\left(-2\sqrt{(1 + (\frac{L}{M})^2} + 4 - \frac{2M}{\sqrt{L^2 + M^2}}\right) \qquad (32)$$

3.1.1.2. The First Ampere Law Term from Branch 1 to 6

Applying integral (2a) gives $d^2\vec{F}_{1\to 6,a} = I^2 M \int_{x_1}\int_{x_2} \frac{2dx_1dx_2}{((x_2-x_1)^2+M^2)^{\frac{3}{2}}}$ (33)

Following the example of Eq.(19) above, the result may straightforwardly be written

$$d^2\vec{F}_{1\to 6,a} = I^2(4\sqrt{(1+(\frac{L}{M})^2} - 4) \quad (34)$$

3.1.1.3. Both Ampere Law Term from Branch 1 to 6

By summing the now the two contributions above, Eq.(32) and (34) respectively, one attains the total Ampère force from branch 1 to 6 (excluding the effects due to the current source)

$$d^2\vec{F}_{1\to 6} = I^2(2\sqrt{(1+(\frac{L}{M})^2} - \frac{2M}{\sqrt{L^2+M^2}}) \quad (35)$$

3.1.2. Integral from branch 2 to 7

In this case

$d\bar{s}_1 = (0, dy_1, 0)$ (36)

$d\bar{s}_2 = (0, -dy_2, 0)$ (37)

$\bar{r} = (-L, y_2 - y_1, 0)$ (38)

$r = \sqrt{(y_2-y_1)^2 + L^2}$ (39)

Performing the integration, y_1 goes from 0 to N and y_2 goes from N to M.

3.1.2.1. The Second Ampere Law Term from Branch 2 to 7

Applying integral (2b)

gives $\vec{F}_{2\to 7,b} = I^2 \int_{y_1=0_1}^{N}\int_{y_2=N}^{M} \frac{3(-(y_2-y_1)^3)dy_1dy_2}{((y_2-y_1)^2+L^2)^{\frac{5}{2}}}$ (40)

In order to solve the integration, the formula

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ (11) [13]

has to be applied, preferably by first integrating one step and rearranging the terms:

$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}$ (12)

Defining $f = ((y_2-y_1)^2 + L^2)$ (41)

Having a term $f^{-\frac{5}{2}}$, it seems reasonable to search for a primitive function $f^{-\frac{3}{2}}$, which after differentiation with respect to x_1 (the order of integration arbitrarily chosen) gives

$-\frac{3}{2}f^{-\frac{5}{2}}\frac{df}{dy_1}$ (42)

Since $\frac{df}{dy_1} = -2(y_2 - y_1)$ (43)

expression (42) develops to $\frac{3(y_2-y_1)}{((x_2-x_1)^2+M^2)^{\frac{5}{2}}}$ (44)

which appears to be equivalent to an immense factor of the integrand.
The remaining factor is $-(y_2 - y_1)$ (45)

Letting the term $\frac{-3(y_2 - y_1)}{((y_2 - y_1)^2 + L^2)^{\frac{5}{2}}}$ be equivalent to the variable $\frac{dv}{dx}$ in the formula (10) above, $u = (y_2 - y_1)^2$ (46)

The integral (40) may now be identified with respective parts of Equation (11).
Hence,

$$\vec{F}_{2\to 7,b} = I^2 \int_{y_1=0}^{N} \int_{y_2=N}^{M} \frac{3(-(y_2 - y_1)^3)dy_1 dy_2}{((y_2 - y_1)^2 + L^2)^{\frac{5}{2}}} =$$

$$\vec{F}_{2\to 7,b} = I^2 \int_{y_1=0}^{N} \int_{y_2=N}^{M} \frac{3(-(y_2 - y_1))}{((y_2 - y_1)^2 + L^2)^{\frac{5}{2}}} (y_2 - y_1)^2 dy_2 dy_1 =$$

$$I^2(-((y_2-N)^2/((y_2-N)^2+L^2)^{\frac{3}{2}})_{y=N}^{M} + (y_2{}^2/(y_2{}^2+L^2)^{\frac{3}{2}})_{y_2=N}^{M})-$$

$$\int_{y=0_1}^{N} \int_{y_2=N}^{M} \frac{-1}{((y_2 - y_1)^2 + L^2)^{\frac{3}{2}}} (-(2(y_2 - y_1))dy_2 dy_1) \quad (47)$$

The two right hand terms must be solved separately.
The second term may first be integrated one step with respect to y_1. Thereafter it remains only integrating one step more for both terms.

Having a term $f^{-\frac{3}{2}}$, it seems reasonable to search for a primitive function $f^{-\frac{1}{2}}$, which after differentiation with respect to y_1 (the order of integration arbitrarily chosen) gives

$$-\frac{1}{2} f^{-\frac{3}{2}} \frac{df}{dy_1} \quad (48)$$

Since $\frac{df}{dy_1} = -2(y_2 - y_1)$ (49)

expression (48) develops to $f^{-\frac{3}{2}}(y_2 - y_1)$ (49)

which also may be expressed as $\frac{(y_2 - y_1)}{((y_2 - y_1)^2 + L^2)^{\frac{3}{2}}}$ (50)

Hence, the primitive function to the second term is $f^{-\frac{1}{2}} = \frac{1}{\sqrt{((y_2 - y_1)^2 + L^2}}$ (51)

Now it is possible to further simplify Eq. (47) by inserting the result (51).

$$I^2(\int_{y_2=N}^{M} dy_2(-((y_2-y_1)^2/((y_2-y_1)^2+L^2)^{3/2} - \frac{1}{\sqrt{(y_2-y_1)^2+L^2}})_{y_1=0}^{N}) \quad (52)$$

Inserting the values of y_1 into (52) now gives:

$$\vec{F}_{2\to 7,b} = I^2 \int_{y_2=N}^{M} \left(\frac{-(y_2-N)^2}{((y_2-N)^2+L^2)^{\frac{3}{2}}} + \frac{{y_2}^2}{({y_2}^2+L^2)^{\frac{3}{2}}} - \frac{2}{\sqrt{(y_2-N)^2+L^2}} + \frac{2}{\sqrt{{y_2}^2+L^2}} \right) dy_2$$

(53)

In order to solve the integration of ***the two first terms***, the formula

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad (10) \qquad [13]$$

has to be applied, preferably by first integrating one step and rearranging the terms:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx} \qquad (11)$$

Defining $f = ((y_2-N)^2+L^2)$ (54) for the first term, the second term will be treated in the same way by setting $N=0$

Having a term $f^{-\frac{3}{2}}$, it seems reasonable to search for a primitive function $f^{-\frac{1}{2}}$, which after differentiation with respect to x_1 (the order of integration arbitrarily chosen) gives

$$-\frac{1}{2}f^{-\frac{3}{2}}\frac{df}{dy_2} \qquad (55)$$

Since $\frac{df}{dy_2} = 2(y_2-N)$ (56)

expression (55) develops to $-\frac{(y_2-N)}{((y_2-N)^2+L^2)^{\frac{3}{2}}}$ (57)

which appears to be equivalent to a dominant factor of the integrand.
The remaining factor is $-(y_2-y_1)$ (58)

Letting the term $\frac{-(y_2-N)}{((y_2-N)^2+L^2)^{\frac{3}{2}}}$ be equivalent to the variable $\frac{dv}{dx}$ in the formula (10) above, $u=(y_2-y_1)$ (59)

Hence, the contribution to Eq.(53) from ***the two first terms*** will be:

=

$$\vec{F}_{2\to 7,b}(twofirstterms) = I^2 \int_{y_2=N}^{M} \left(\frac{-(y_2-N)}{((y_2-N)^2+L^2)^{\frac{3}{2}}}(y_2-N) + \frac{y_2}{({y_2}^2+L^2)^{\frac{3}{2}}} y_2 \right) dy_2 =$$

$$I^2\left((y_2-N)/((y_2-N)^2+L^2)^{1/2}\right)_{y_2=N}^{M} - I^2\left((y_2)/({y_2}^2+L^2)^{1/2}\right)_{y_2=N}^{M} - \int_{y_2=M}^{N} \frac{dy_2}{\sqrt{(y_2-N)^2+L^2}} + \int_{y_2=N}^{M} \frac{dy_2}{\sqrt{{y_2}^2+L^2}} =$$

$$I^2\left(\frac{M-N}{\sqrt{(M-N)^2+L_2}} - \frac{M}{\sqrt{M^2+L^2}} + \frac{N}{\sqrt{N^2+L^2}}\right) - \int_{y_2=M}^{N} \frac{dy_2}{\sqrt{(y_2-N)^2+L^2}} + \int_{y_2=N}^{M} \frac{dy_2}{\sqrt{{y_2}^2+L^2}} \qquad (60)$$

which through usage of the integral $\int \frac{dx}{(x^2 \pm a^2)^{\frac{1}{2}}} = \ln\left| x + (x^2 \pm a^2)^{\frac{1}{2}} \right|$ [13b] (61)

gives

$$\vec{F}_{2\to 7,b}(twofirstterms) = I^2 \left(\frac{M-N}{\sqrt{(M-N)^2 + L_2}} - \frac{M}{\sqrt{M^2+L^2}} + \frac{N}{\sqrt{N^2+L^2}} \right) +$$

$$I^2 \left((-\ln(y_2 - N + \sqrt{(y_2-N)^2 + L^2}) + \ln(y_2 + \sqrt{y_2{}^2 + L^2}) \right)_{y_2=N}^{M} \quad =(62)$$

It remains then to integrate ***the two two last terms*** of Eq. (53).
which in the case of Eq. (53) gives for ***the two two last terms***

$$\vec{F}_{2\to 7,b}(twolastterms) = I^2 \int_{y_2=N}^{M} \left(\frac{-2}{\sqrt{(y_2-N)^2+L^2}} + \frac{2}{\sqrt{y_2{}^2+L^2}} \right) dy_2 =$$

$$I^2 \left((-2\ln(y_2 - N + \sqrt{(y_2-N)^2 + L^2}) + 2\ln(y_2 + \sqrt{y_2{}^2 + L^2}) \right)_{y_2=N}^{M} \quad (63)$$

Summing the results from the two first terms to thte two last terms of Eq.(53) accordingly gives

$$\vec{F}_{2\to 7,b} = I^2 \left(\frac{M-N}{\sqrt{(M-N)^2 + L_2}} - \frac{M}{\sqrt{M^2+L^2}} + \frac{N}{\sqrt{N^2+L^2}} \right) +$$

$$I^2 \left((-3\ln(y_2 - N + \sqrt{(y_2-N)^2 + L^2}) + 3\ln(y_2 + \sqrt{y_2{}^2 + L^2}) \right)_{y_2=N}^{M} \quad (64)$$

Performing the evaluation of the two last terms consisting of primitive functions, between the given boarders, gives

$$\vec{F}_{2\to 7,b} =$$

$$I^2 \left(\frac{M-N}{\sqrt{(M-N)^2 + L_2}} - \frac{M}{\sqrt{M^2+L^2}} + \frac{N}{\sqrt{N^2+L^2}} - 3\ln\frac{M-N+\sqrt{(M-N)^2+L^2}}{L} \right) + 3\ln\frac{M+\sqrt{M^2+L^2}}{N+\sqrt{N^2+L^2}} \Big)$$

(65)

3.1.2.2. The First Ampere Law Term from Branch 2 to 7

Applying integral (2a) gives $\vec{F}_{2\to 7,a} = I^2 \int_{y_1=0}^{N} \int_{y_2=N}^{M} \frac{2(y_2 - y_1) dy_1 dy_2}{((y_2-y_1)^2 + L^2)^{\frac{3}{2}}}$ (66)

Having a term $f^{-\frac{3}{2}}$, it seems reasonable to search for a primitive function $f^{-\frac{1}{2}}$, which after differentiation with respect to y_1 (the order of integration arbitrarily chosen) gives

$$-\frac{1}{2}f^{-\frac{3}{2}}\frac{df}{dy_1} \qquad (48)$$

Since $\frac{df}{dy_1} = -2(y_2 - y_1)$ (49)

expression (48) develops to $f^{-\frac{3}{2}}(y_2 - y_1)$ (49)

which also may be expressed as $\frac{(y_2 - y_1)}{((y_2 - y_1)^2 + L^2)^{\frac{3}{2}}}$ (50)

Hence, integrating with respect to y_1 gives

$$\vec{F}_{2\to 7,a} = I^2 \int_{y_2=N}^{M} (2/\sqrt{(y_2-y_1)^2+L^2})_{y=0}^{N} \qquad (67)$$

which develops to $\vec{F}_{2\to 7,a} = \quad I^2 \int_{y_2=N_1}^{M} (\frac{2}{\sqrt{(y_2-N)^2+L^2}} - \frac{2}{\sqrt{(y_{2_2}{}^2 + L^2}})$ (68)

Using again integral (61) gives the result

$$\vec{F}_{2\to 7,a} = I^2((2\ln(y_2-N+\sqrt{(y_2-N)^2+L^2}))_{y_2=N}^{M} - 2\ln(y_2+\sqrt{y_2{}^2+L^2})_{y_2=N}^{M})$$

(69)

Evaluation gives $\vec{F}_{2\to 7,a} = I^2(2\ln\frac{M-N+\sqrt{(M-N)^2+L^2}}{L} - 2\ln\frac{M+\sqrt{M^2+L^2}}{N+\sqrt{N^2+L^2}})$

(70)

3.1.2.3. Both Ampere Law Term from Branch 2 to 7

By summing the now the two contributions above, Eq.(65) and (70) respectively, one attains the total Ampère force from branch 2 to 7

$\vec{F}_{2\to 7,b} =$

$$I^2(\frac{M-N}{\sqrt{(M-N)^2+L_2}} - \frac{M}{\sqrt{M^2+L^2}} + \frac{N}{\sqrt{N^2+L^2}} - \ln\frac{M-N+\sqrt{(M-N)^2+L^2}}{L}) + \ln\frac{M+\sqrt{M^2+L^2}}{N+\sqrt{N^2+L^2}})$$

(71)

3.1.3. Integral from branch 10 to branch 5

The integral from branch 10 to branch 5 has been chosen directly after the case with branch 2 to branch 7, since they show great similarities. Please compare with section 3.1.2, Eq. (36) to (39).

In this case

$d\vec{s}_1 = (0,-dy_1,0)$ (72)

$d\vec{s}_2 = (0,dy_2,0)$ (73)

$\vec{r} = (L, y_2 - y_1, 0)$ (74)

$r = \sqrt{(y_2 - y_1)^2 + L^2}$ (75)

Performing the integration y_1 goes from 0 to N and y_2 goes from N to M.

As can be seen, the sign has been changed in the first two equations and in the third one the sign before L has been changed. In writing eq. (2) for this case, all these changes appear in pairs, and therefore, cancel.

Therefore, one is able to write down the final result without performing any calculations, i.e.

$\vec{F}_{10\to 5} = \vec{F}_{2\to 7} =$

$$I^2\left(\frac{M-N}{\sqrt{(M-N)^2 + L_2}} - \frac{M}{\sqrt{M^2+L^2}} + \frac{N}{\sqrt{N^2+L^2}} - \ln\frac{M-N+\sqrt{(M-N)^2+L^2}}{L}\right) + \ln\frac{M+\sqrt{M^2+L^2}}{N+\sqrt{N^2+L^2}})$$

(76)

That can also be expressed by saying that the integral from branch 10 to branch 5 is symmetric with respect to the integral from branch 2 to branch 7.

3.1.4. Integral from branch 1 to branch 5 (excluding the effect due to the current source in branch 1 –please see chapter 5.3.3)

In this case

$d\vec{s}_1 = (dx_1, 0, 0)$ (77)

$d\vec{s}_2 = (0, dy_2, 0)$ (78)

$\vec{r} = (L - x_1, y_2, 0)$ (79)

$r = \sqrt{(L - x_1)^2 + {y_2}^2}$ (80)

Performing the integration, x_1 goes from 0 to L and y_2 too, goes from N to M

3.1.4.1. The Second Ampere Law Term from Branch 1 to 5 (excluding the effect due to the current source in branch 1)

Applying integral (2b)

gives $\vec{F}_{1\to 5,b} = I^2 \int_{x_1=0_1}^{L} \int_{y_2=N}^{M} \frac{3{y_2}^2(L-x_1)dx_1 dy_2}{((L-x_1)^2 + {y_2}^2)^{\frac{5}{2}}}$ (81)

Defining $f = ((L - x_1)^2 + {y_2}^2)$ (82)

having a term $f^{-\frac{5}{2}}$, it seems reasonable to search for a primitive function $f^{-\frac{3}{2}}$, which after differentiation with respect to x_1 (the order of integration arbitrarily chosen) gives

$-\frac{3}{2} f^{-\frac{5}{2}} \frac{df}{dx_1}$ (14)

Since $\frac{df}{dx_1} = -2(L - x_1)$ (83)

expression (14) develops to $\frac{3(L-x_1)}{((L-x_1)^2 + {y_2}^2)^{\frac{5}{2}}}$ (84)

which appears to be equivalent to a dominant factor of the integrand.

The remaining factor is $(L - x_1)^2$ (85)

Letting the term $\frac{3(L-x_1)}{((x_2-x_1)^2+M^2)^{2\frac{5}{2}}}$ be equivalent to the variable $\frac{dv}{dx}$ in the formula (11)

above, $u = y_2{}^2$ (86)

The integral (81) may now be identified with respective parts of Equation (12).
Hence,

$$\bar{F}_{1\to 5,b} = I^2 \int_{x_1=0_1}^{L} \int_{y_2=N}^{M} \frac{3y_2{}^2(L-x_1)dx_1dy_2}{((L-x_1)^2+y_2{}^2)^{\frac{5}{2}}} =$$

$$I^2 \int_{y_2=N}^{M} y_2{}^2 dy_2 (1/((L-x_1)^2+y_2{}^2)^{3/2})_{x_1=0}^{L} =$$

$$I^2 (\int_{y_2=N}^{M} \frac{dy_2}{y_2} - \int_{y2=N}^{M} \frac{y_2{}^2 dy_2}{(L^2+y_2{}^2)^{\frac{3}{2}}}) \quad (87)$$

The first term easily develops to $I^2 (\ln y_2)_{y_2=N}^{M} = I^2(\ln\frac{M}{N})$ (88)

The second term may be solved if assuming that

having a term $f^{-\frac{3}{2}}$, it seems reasonable to search for a primitive function $f^{-\frac{1}{2}}$, which after differentiation with respect to x_1 (the order of integration arbitrarily chosen) gives

$$-\frac{1}{2} f^{-\frac{3}{2}} \frac{df}{dy_2} \quad (55)$$

Since $\frac{df}{dy_2} = 2y_2$ (89)

expression (55) develops to $-\frac{y_2}{(L^2+y_2{}^2)^{\frac{3}{2}}}$ (90)

which appears to be equivalent to a dominant factor of the integrand.
The remaining factor is y_2 (91)

Letting the term $-\frac{y_2}{(L^2+y_2{}^2)^{\frac{3}{2}}}$ be equivalent to the variable $\frac{dv}{dx}$ in the formula (11) above,

$u = y_2$ (92)

The second term thus becomes $I^2((y_2/\sqrt{y_2{}^2+L^2})_{y_2=N}^{M} - \int_{y_2=N}^{M} \frac{dy_2}{\sqrt{y_2{}^2+L^2}})$ (93)

Using the integral (61) the right term can be rewritten and

Eq. (93) now develops to $I^2(\frac{M}{\sqrt{M^2+L^2}} - \frac{N}{\sqrt{N^2+L^2}} - \ln(\frac{M+\sqrt{M^2+L^2}}{N+\sqrt{N^2+L^2}})$ (94)

Adding the two terms (88) and (93), thereby taking account that gives the total integral

$$\bar{F}_{1\to 5,b} = I^2(\ln\frac{M}{N} + \frac{M}{\sqrt{M^2+L_2}} - \frac{N}{\sqrt{N^2+L^2}} - \ln\frac{M+\sqrt{M^2+L^2}}{N+\sqrt{N^2+L^2}}) \quad (95)$$

3.1.4.2. The First Ampere Law Term from Branch 1 to 5 (excluding the effect due to the current source in branch 1)

In this case there will be no integral (2a), since $d\bar{s}_1$ is perpendicular to $d\bar{s}_2$.

3.1.4.3 Both Ampere Law Terms from Branch 1 to 5 (excluding the effect due to the current source in branch 1)

Due to the statement in 3.1.4.2 Eq. (95) expresses also the total integral and hence,

$$\vec{F}_{1\to5} = I^2(\ln\frac{M}{N} + \frac{M}{\sqrt{M^2+L_2}} - \frac{N}{\sqrt{N^2+L^2}} - \ln\frac{M+\sqrt{M^2+L^2}}{N+\sqrt{N^2+L^2}}) \qquad (96)$$

3.1.5. Integral from branch 1 to branch 7 (excluding the effect due to the current source in branch 1-please see chapter 5.3.2)

The integral from branch 1 to branch 7 has been chosen directly after the case with branch 1 to branch 5, since they show great similarities. Please compare with section 3.1.4, Eq. (77) to (80).

In this case

$d\bar{s}_1 = (dx_1,0,0)$ (97)

$d\bar{s}_2 = (0,-dy_2,0)$ (98)

$\vec{r} = (-x_1, y_2, 0)$ (99)

$r = \sqrt{x_1^2 + y_2^2}$ (100)

Performing the integration, x_1 goes from 0 to L and y_2 too, goes from N to M

In this case

3.1.5.1. The Second Ampere Law Term from Branch 1 to 7 (excluding the effect due to the current source in branch 1)

Applying integral (2b)

gives $$\vec{F}_{1\to7,b} = I^2 \int_{x_1=0_1}^{L}\int_{y_2=N}^{M} \frac{3y_2(-x_1)(-y_2)dx_1dy_2}{((-x_1)^2+y_2^2)^{\frac{5}{2}}} = I^2 \int_{x_1=0_1}^{L}\int_{y_2=N}^{M} \frac{3y_2^2x_1dx_1dy_2}{((-x_1)^2+y_2^2)^{\frac{5}{2}}} \qquad (101)$$

However, integrating a function of x_1 between 0 and L is equal to integrating the same function with argument $L-x_1$ along the same interval, as is being done in Eq. (81), due to the case with branch 1 affecting branch 5. Hence,

$$\vec{F}_{1\to7,b} = \vec{F}_{1\to5,b} = I^2(\ln\frac{M}{N} + \frac{M}{\sqrt{M^2+L_2}} - \frac{N}{\sqrt{N^2+L^2}} - \ln\frac{M+\sqrt{M^2+L^2}}{N+\sqrt{N^2+L^2}}) \qquad (102)$$

3.1.5.2. The First Ampere Law Term from Branch 1 to 7 (excluding the effect due to the current source in branch 1)

In this case there will be no integral (2a), since $d\bar{s}_1$ is perpendicular to $d\bar{s}_2$, just similar to the case with branch 1 affecting branch 5, described in section 3.1.4.2.

3.1.5.3 Both Ampere Law Terms from Branch 1 to 7 (excluding the effect due to the current source in branch 1)

Due to the statement in 3.1.5.2 Eq. (102) expresses also the total integral and hence,

$$\vec{F}_{1\to 7} = I^2(\ln\frac{M}{N} + \frac{M}{\sqrt{M^2+L_2}} - \frac{N}{\sqrt{N^2+L^2}} - \ln\frac{M+\sqrt{M^2+L^2}}{N+\sqrt{N^2+L^2}}) \qquad (103)$$

3.1.6. Integral from branch 2 to 6

In this case

$d\vec{s}_1 = (0, dy_1, 0)$ (104)

$d\vec{s}_2 = (-dx_2, 0, 0)$ (105)

$\vec{r} = (x_2 - L, M - y_1, 0)$ (106)

$r = \sqrt{(x_2 - L)^2 + (M - y_1)^2}$ (107)

Performing the integration, y_1 goes from 0 to N and x_2 goes from 0 to L.

3.1.6.1. The Second Ampere Law Term from Branch 2 to 6

gives $\vec{F}_{2\to 6,b} = I^2 \int\limits_{y_1=0_1}^{N} \int\limits_{x_2=0}^{L} \frac{3(M-y_1)^2(-(x_2-L)dy_1 dx_2}{(((M-y_1)^2 + (-(x_2-L))^2)^{\frac{5}{2}}}$ (108)

Defining $f = ((M-y_1)^2 + (-(x_2-L))^2)$ (109)

having a term $f^{-\frac{5}{2}}$, it seems reasonable to search for a primitive function $f^{-\frac{3}{2}}$, which after differentiation with respect to x_1 (the order of integration arbitrarily chosen) gives

$-\frac{3}{2} f^{-\frac{5}{2}} \frac{df}{dx_2}$ (110)

Since $\frac{df}{dx_2} = 2(x_2 - L)$ (111)

expression (108) develops to $\frac{-3(x_2-L)}{((M-y_1)^2 + (x_2-L)^2)^{\frac{5}{2}}}$ (112)

which appears to be equivalent to a dominant factor of the integrand.

The remaining factor is $(M-y_1)^2$ (113)

Letting the term $\frac{-3(x_2-L)}{((M-y_1)^2 + (-(x_2-L))^2)^{\frac{5}{2}}}$ be equivalent to the variable $\frac{dv}{dx}$ in the formula (11) above, $u = (M-y_1)^2$ (114)

The integral (108) may now be identified with respective parts of Equation (11).

Hence,

$$\vec{F}_{2\to 6,b} = I^2 \int\limits_{y_1=0}^{N} ((M-y_1)^2 / ((M-y_1)^2 + (-(L-x_2))^2)^{3/2})_{x_2=0}^{L} dy_1 =$$

$$\vec{F}_{2\to 6,b} = I^2 \int\limits_{y_1=0}^{N} (M-y_1)^2 (\frac{1}{(M-y_1)^{\frac{3}{2}}} - \frac{1}{(L^2 + ((M-y_1)^2)^{\frac{3}{2}}}) dy_1 \qquad (115)$$

The first term will be integrated first since it is most simply integrateble. It may be written

$$\vec{F}_{2\to 6,b}(firstterm) = I^2 \int\limits_{y_1=0}^{N} \frac{1}{\sqrt{M-y_1}} = (-\ln(M-y_1))_{y_1=0}^{N} = -I^2 \ln\frac{M-N}{M} \qquad (116)$$

The analysis of the second term will develop as follows, thereby using the partial integral formula (11)

$$d^2\vec{F}_{2\to 6,b}(\sec ondterm) = I^2(-((M-y_1)/\sqrt{(M-y_1)^2+L^2})_{y_1=0}^{N} - \int_{y_1=0}^{N} \frac{dy_1}{\sqrt{(M-y_1)^2+L^2}})$$

(117)

Applying Eq. (61) upon the last term within this expression and thereafter evaluating accordingly, gives

$$\vec{F}_{2\to 6,b}(\sec ondterm) = I^2(-\frac{M-N}{\sqrt{(M-N)^2+L^2}} + \frac{M}{\sqrt{M^2+L^2}} + \ln\frac{M-N+\sqrt{(M-N)^2+L^2}}{M+\sqrt{(M^2+L^2}})$$

(118)

Adding the two terms (116) and (118), gives the total integral

$$\vec{F}_{2\to 6,b} = I^2(-\ln\frac{M-N}{M} - \frac{M-N}{\sqrt{(M-N)^2+L^2}} + \frac{M}{\sqrt{M^2+L^2}} + \ln\frac{M-N+\sqrt{(M-N)^2+L^2}}{M+\sqrt{(M^2+L^2}})$$

(119)

3.1.6.2. The First Ampere Law Term from Branch 2 to 6

In this case there will be no integral (2a), since $d\vec{s}_1$ is perpendicular to $d\vec{s}_2$, just similar to the case with branch 1 affecting branch 5, described in section 3.1.4.2.

3.1.6.3 Both Ampere Law Terms from Branch 2 to 6

Due to the statement in 3.1.6.2 Eq. (119) expresses also the total integral and hence,

$$\vec{F}_{2\to 6} = I^2(-\ln\frac{M-N}{M} - \frac{M-N}{\sqrt{(M-N)^2+L^2}} + \frac{M}{\sqrt{M^2+L^2}} + \ln\frac{M-N+\sqrt{(M-N)^2+L^2}}{M+\sqrt{(M^2+L^2}})$$

(120)

3.1.7. Integral from branch 10 to 6

The integral from branch 10 to branch 6 has been chosen directly after the case with branch 2 to branch 6, since they show great similarities. Please compare with section 3.1.6, Eq. (104) to (107). In this case

$d\vec{s}_1 = (0,-dy_1,0)$ (121)

$d\vec{s}_2 = (-dx_2,0,0)$ (122)

$\vec{r} = (x_2, M-y_1, 0)$ (123)

$r = \sqrt{x_2^{\,2} + (M-y_1)^2}$ (124)

Performing the integration, y_1 goes from 0 to N and x_2 goes from 0 to L.

3.1.6.1. The Second Ampere Law Term from Branch 10 to 6

Applying integral (2b)

gives

$$\vec{F}_{10\to 6,b} = I^2 \int_{y_1=0_1}^{N} \int_{x_2=0}^{L} \frac{3(M-y_1)^2 x_2 dy_1 dx_2}{(((M-y_1)^2+x_2^{\,2})^{\frac{5}{2}}} \qquad (125)$$

However, integrating a function of x_2 between 0 and L is equal to integrating the same function with argument $L-x_2$ along the same interval, as is being done in Eq. (108), due to the case with branch 2 affecting branch 6. Hence, using Eq. (119) for the result of $\vec{F}_{2\to 6,b}$,

$$\vec{F}_{10\to 6,b}=\vec{F}_{2\to 6,b}=I^2(-\ln\frac{M-N}{M}-\frac{M-N}{\sqrt{(M-N)^2+L^2}}+\frac{M}{\sqrt{M^2+L^2}}+\ln\frac{M-N+\sqrt{(M-N)^2+L^2}}{M+\sqrt{(M^2+L^2}})$$

(126)

3.1.7.2. The First Ampere Law Term from Branch 10 to 6

In this case there will be no integral (2a), since $d\vec{s}_1$ is perpendicular to $d\vec{s}_2$, just similar to the case with branch 1 affecting branch 5, described in section 3.1.4.2.

3.1.7.3 Both Ampere Law Terms from Branch 10 to 6

Due to the statement in 3.1.7.2 Eq. (126) expresses also the total integral and hence,

$$\vec{F}_{10\to 6}=\vec{F}_{2\to 6}=I^2(-\ln\frac{M-N}{M}-\frac{M-N}{\sqrt{(M-N)^2+L^2}}+\frac{M}{\sqrt{M^2+L^2}}+\ln\frac{M-N+\sqrt{(M-N)^2+L^2}}{M+\sqrt{(M^2+L^2}})$$

(127)

The sum of all the contributions thus far (from parts of the bridge far away from each others) is: (thereby using Wesley's notation [4]):

$$F'=2I^2(\sqrt{1+(\frac{L}{M})^2}-\ln\frac{M-N}{M}-\ln\frac{N}{L}-\ln(1+\sqrt{1+(\frac{L}{M})^2}) \qquad (128)$$

This expression has to be added to the contribution from the parts of the bridge that are in close contact to each others, i.e. $2\to 5$ and $10\to 7$ respectively, Eq. (306).

3.1.8. Integral from branch 10 to 7

This case is more difficult to treat than the preceding ones, since here these two branches (like 2 to 5) come close to each others and therefore they can not be treated as thin conductors consisting of a line. This means that volume integrals will have to be used, involving principally the three Cartesian variables of both branches. The integral formulas are defined in Eq. (1). Since the y component of the force is requested, the scalar product has to be taken with $\vec{u}_y=(0,1,0)$ (129)

While $\vec{r}=(x_2-x_1,y_2-y_1,z_2-z_1)$ (130)

as in the preceding cases treated above,

since in all cases the y component of the force is being analysed, for simplicity all the results due to Eq. (1) and (2) treated below will mean the y component.

Thus, instead of always writing $d^6\vec{F}\bullet\vec{u}_y=........$ it will simpler be written $d^6\vec{F}=........$

In this case

$d\vec{s}_1=(0,dy_1,0)$ (131)

$d\vec{s}_2=(0,dy_2,0)$ (132)

$\vec{r}=(x_2-x_1,y_2-y_1,z_2-z_1)$ (133)

$r=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ (134)

By practical reasons the two last terms below the root sign will be called

$A^2=(x_2-x_1)^2+(z_2-z_1)^2$ (135)

3.1.8.2. The First Ampere Law Term from Branch 10 to 7

Using Eq. (1a), this leads to

$$\vec{F}_{10\to 7,a} = J^2 \int_{z_1=0}^{t} dz_2 \int_{y_2=N}^{M} dy_2 \int_{x_2=0}^{w} dx_2 \int_{z_1=0}^{t} dz_1 \int_{y_1=0}^{N} dy_1 \int_{x_1=0}^{w} dx_1 \frac{-2(y_2-y_1)}{r^3} \quad (136)$$

Integrating first the integrand first with respect to y_2 gives the primitive function

$$(2/r)_{y_2=N}^{M} = \frac{2}{\sqrt{(M-y_1)^2+(x_2-x_1)^2+(z_2-z_1)^2}} - \frac{2}{\sqrt{(N-y_1)^2+(x_2-x_1)^2+(z_2-z_1)^2}}$$

(137)

Integrating in the next step with respect to y_1 gives the result

$$\vec{F}_{10\to 7,a} = J^2 \int_{z_1=0}^{t} dz_2 \int_{x_2=0}^{w} dx_2 \int_{z_1=0}^{t} dz_1 \int_{x_1=0}^{w} dx_1 (-2\log\frac{M-N+\sqrt{(M-N)^2+A^2}}{M+\sqrt{M^2+A^2}} + 2\log\frac{A}{N+\sqrt{N^2+A^2}})$$

(138)

The remaining four integrals will follow.

3.1.8.3. The Second Ampere Law Term from Branch 10 to 7

Using in this case Eq. (1b)

$$\vec{F}_{10\to 7,b} = J^2 \int_{z_1=0}^{t} dz_2 \int_{y_2=N}^{M} dy_2 \int_{x_2=0}^{w} dx_2 \int_{z_1=0}^{t} dz_1 \int_{y_1=0}^{N} dy_1 \int_{x_1=0}^{w} dx_1 \frac{3(y_2-y_1)^3}{r^5} \quad (139)$$

3.1.8.3.1. Integrating with respect to y_2

Having a term $f^{-\frac{5}{2}}$, $f = r^2$ (140)

it seems reasonable to search for a primitive function $f^{-\frac{3}{2}}$, which after differentiation with respect to y_2 (the order of integration arbitrarily chosen) gives

$$-\frac{3}{2} f^{-\frac{5}{2}} \frac{df}{dy_2} \quad (141)$$

Since $\frac{df}{dy_2} = 2(y_2 - y_1)$ (142)

expression (141) develops to $\dfrac{-3(y_2-y_1)}{((y_2-y_1)^2+A^2)^{\frac{5}{2}}}$ (143)

which appears to be equivalent to a dominant factor of the integrand.

The remaining factor is $(y_2-y_1)^2$ (144)

Letting the term $\dfrac{3(y_2-y_1)}{((y_2-y_1)^2+A^2)^{\frac{5}{2}}}$ be equivalent to the variable $\dfrac{dv}{dx}$ in the formula (10)

$$\frac{dv}{dx} = \frac{3(y_2-y_1)}{((y_2-y_1)^2+A^2)^{\frac{5}{2}}} \quad (145)$$ above,

$$u = (y_2-y_1)^2 \quad (146)$$

Integrating accordingly one step with respect to y_2, thus using the formula for partial integration (11) gives the following result for the primitive function:

$$-(((y_2-y_1)^2/((y_2-y_1)^2+A^2)^{3/2})_{y_2=N}^{M} - \int_{y_2=N}^{M} \frac{2(y_2-y_1)}{((y_2-y_1)^2+A^2)^{\frac{3}{2}}}) \qquad (147)$$

3.1.8.3.2. Integrating with respect to y_1

The last term being $\sim f^{-\frac{3}{2}} = r^3$, it seems reasonable to search for a primitive function $f^{-\frac{1}{2}}$, which after differentiation with respect to y_1 (the order of integration arbitrarily chosen) gives

$$-\frac{1}{2} f^{-\frac{3}{2}} \frac{df}{dy_1} \qquad (148)$$

Since $\frac{df}{dy_1} = 2(y_2 - y_1)$ (149)

expression (143) develops to $-f^{-\frac{3}{2}}(y_2 - y_1)$ (150)

which also may be expressed as $\frac{-(y_2-y_1)}{((y_2-y_1)^2+L^2)^{\frac{3}{2}}}$ (151)

Hence, Eq. (147) develops to

$$\vec{F}_{10\to 7,b} = J^2 \int_{z_1=0}^{t} dz_2 \int_{x_2=0}^{w} dx_2 \int_{z_1=0}^{t} dz_1 \int_{y_1=0}^{N} dy_1 \int_{x_1=0}^{w} dx_1 (-(\frac{(M-y_1)^2}{((M-y_1)^2+A^2)^{\frac{3}{2}}} - \frac{(N-y_1)^2}{((N-y_1)^2+A^2)^{\frac{3}{2}}}) -$$

$$(\frac{2}{\sqrt{((M-y_1)^2+A^2}} - \frac{2}{\sqrt{(N-y_1)^2+A^2}})) \qquad (152)$$

By practical reasons the next integration step, using y_1 , one may solve each term separately, calling them by Roman numbers I, II, III and IV respectively.
Using the partial integration formula (11) gives:

$$\int_{y_1=0}^{N} \frac{-(M-y_1)^2}{((M-y_1)^2+A^2)^{\frac{3}{2}}} dy_1 = -((M-y_1)/\sqrt{(M-y_1)^2+A^2})_{y_1=0}^{N} + \int_{y_1=0}^{N} \frac{-dy_1}{\sqrt{(M-y_1)^2+A^2}}$$

(153)

which may further be developed and accordingly, evaluated, thus attaining

$$-\frac{M-N}{\sqrt{(M-N)^2+A^2}} + \frac{M}{\sqrt{M^2+A^2}} + \log\frac{M-N+\sqrt{(M-N)^2+A^2}}{M+\sqrt{M^2+A^2}} \qquad (154)$$

Performing in the similar way with term II of Eq. (152) gives accordingly:

$$-\frac{N}{\sqrt{N^2+A^2}} - \log\frac{A}{N+\sqrt{N^2+A^2}} \qquad (155)$$

Term III can be straightforwardly integrated:

$$\int_{y_1=0}^{N} \frac{-2dy_1}{\sqrt{((M-y_1)^2+A^2}} = 2\log\frac{M-N+\sqrt{(M-N)^2+A^2}}{M+\sqrt{M^2+A^2}} \qquad (156)$$

Following this example term IV accordingly becomes

$$\int_{y_1=0}^{N} \frac{2dy_1}{\sqrt{((N-y_1)^2+A^2}} = 2\log\frac{A}{N+\sqrt{N^2+A^2}} \qquad (152)$$

Summing now the contributions (138), (154), (155), (156) and (157) will give the total result due to both Ampère Law terms after having integrated with respect to both the y vatriables:

$$\vec{F}_{10\to 7} = J^2 \int_{z_1=0}^{t} dz_2 \int_{x_2=0}^{w} dx_2 \int_{z_1=0}^{t} dz_1 \int_{x_1=0}^{w} dx_1 \Big(\frac{M}{\sqrt{M^2+A^2}} - \frac{M-N}{\sqrt{(M-N)^2+A^2}} - \frac{N}{\sqrt{N^2+A^2}} +$$

$$\ln\frac{M-N+\sqrt{(M-N)^2+A^2}}{M+\sqrt{M^2+A^2}} - \ln\frac{A}{N+\sqrt{N^2+A^2}}\Big) \qquad (158)$$

If the contribution from the branches 2-5 is added, we get twice the expression, which is simultaneously what Wesley [2] has attained. We begin with 10-7 (all variables beginning with zero)

Wesley [2] in his solution makes use of the mean value theorem for integrals for $w \to 0$, $t \to 0$, or for $R \to 0$, where $I = Jwt$ (159)

This makes it possible to simplify Eq. (158), thereby using (135), to

$$\vec{F}_{10\to 7} \cong I^2\Big(-1-\log\frac{M}{M-N}+\log 2N\Big) - J^2 \int_{z_2=0}^{t} dz_2 \int_{x_2=0}^{w} dx_2 \int_{z_1=0}^{t} dz_1 \int_{x_1=0}^{w} dx_1 \ln((x_2-x_1)^2+(z_2-z_1)^2)$$

(160)

By practical reasons the last terms will get a name according to:

$$\vec{F}'' \cong -J^2 \int_{z_2=0}^{t} dz_2 \int_{x_2=0}^{w} dx_2 \int_{z_1=0}^{t} dz_1 \int_{x_1=0}^{w} dx_1 \ln((x_2-x_1)^2+(z_2-z_1)^2) \qquad (161)$$

Now remains four integration steps, which will cause some tedious calculations, which for the sake of clarity will be shown here.

3.1.8.3.3. Integrating with respect to x_2

Wanting to solve the integral using partial integration techniques, which have appeared very useful thus far, leads to the primitive function

$(x_2-x_1)\log((x_2-x_1)^2+(z_2-z_1)^2)$ (162)

Differentiating this expression gives:

$$\frac{\partial}{\partial x_2}\big((x_2-x_1)\ln((x_2-x_1)^2+(z_2-z_1)^2)\big) =$$

$$\ln((x_2-x_1)^2+(z_2-z_1)^2) + \frac{2(x_2-x_1)}{(x_2-x_1)^2+(z_2-z_1)^2} \qquad (163)$$

Before integrating with respect to x_2, this result may favourably be used in order to begin solving the integral according to Eq. (161) :

$$\vec{F}'' \cong -J^2 \int_{z_2=0}^{t} dz_2 \int_{z_1=0}^{t} dz_1 \int_{x_1=0}^{w} dx_1 \Big((x_2-x_1)\ln((x_2-x_1)^2+(z_1-z_2)^2)\Big)_{x_2=0}^{w} +$$

$$+J^2 \int_{z_2=0}^{t} dz_2 \int_{x_2=0}^{w} dx_2 \int_{z_1=0}^{t} dz_1 \int_{x_1=0}^{w} dx_1 \frac{2(x_2-x_1)^2}{(x_2-x_1)^2+(z_2-z_1)^2} \qquad (164)$$

$$\vec{F}''(firstterm) \cong -J^2 \int_{z_2=0}^{t} dz_2 \int_{z_1=0}^{t} dz_1 \int_{x_1=0}^{w} dx_1 ((w-x_1)\ln((w-x_1)^2+(z_2-z_1)^2)+x_1\ln(x_1{}^2+(z_2-z_1)^2))$$

(165)

The second term may be developed as follows:

$$\vec{F}''(\sec ondterm) = +J^2 \int_{z_2=0}^{t} dz_2 \int_{x_2=0}^{w} dx_2 \int_{z_1=0}^{t} dz_1 \int_{x_1=0}^{w} dx_1 \frac{2(x_2-x_1)^2}{(x_2-x_1)^2+(z_2-z_1)^2} =$$

$$+J^2 \int_{z_2=0}^{t} dz_2 \int_{x_2=0}^{w} dx_2 \int_{z_1=0}^{t} dz_1 \int_{x_1=0}^{w} dx_1 (2-\frac{2(z_2-z_1)^2}{(x_2-x_1)^2+(z_2-z_1)^2}) \qquad (166)$$

The first term of that expression is simply solved; the second requires the usage of the integration formula $\int \frac{dx}{a^2+b^2x^2} = \frac{1}{ab}\arctan\frac{bx}{a}$ [14] (167)

Hence,

$$\vec{F}''(\sec ondterm) =$$

$$+J^2 \int_{z_2=0}^{t} dz_2 \int_{z_1=0}^{t} dz_1 \int_{x_1=0}^{w} dx_1 (2w-2((z_2-z_1)\arctan(x_2-x_1)/(z_2-z_1))_{x_2=0}^{w}) \qquad (168)$$

Now the results due to development of both terms of the integral Eq. (164) above may be gathered through adding Eq.(165) to Eq. (168):

$$\vec{F}'' \cong -\frac{1}{2}J^2 \int_{z_2=0}^{t} dz_2 \int_{z_1=0}^{t} dz_1 \int_{x_1=0}^{w} dx_1 ((w-x_1)\ln((w-x_1)^2+(z_2-z_1)^2)+x_1\ln(x_1{}^2+(z_2-z_1)^2)-2w+$$

$$+2((z_2-z_1)(\arctan\frac{w-x_1}{z_2-z_1}+\arctan\frac{x_1}{z_2-z_1})) \qquad (169)$$

(x being used as a local variable during each of the following integrations)

3.1.8.3.4. Integrating with respect to x_1

Now there are five terms to be integrated. They will best be treated separately. Due to the relatively simple shape, the second term will first be treated. Thus,

$$\vec{F}''(\sec ondterm) \cong -\frac{1}{2}J^2 \int_{z_2=0}^{t} dz_2 \int_{z_1=0}^{t} dz_1 \int_{x_1=0}^{w} dx_1 x_1 \ln(x_1{}^2+(z_2-z_1)^2) \qquad (170)$$

The formula (11) for partial integration will favourably be used in this case. Hence,

$$\int x_1 \ln(x_1{}^2+(z_2-z_1)^2 = \frac{1}{2}x_1{}^2 \ln(x_1{}^2+(z_2-z_1)^2) - \int \frac{x_1{}^3 dx_1}{x_1{}^2+(z_2-z_1)^2} \qquad (171)$$

While $-\int \frac{x_1{}^3 dx_1}{x_1{}^2+(z_2-z_1)^2} = -\int x_1 dx_1 + (z_2-z_1)^2 \int \frac{x_1 dx_1}{x_1{}^2+(z_2-z_1)^2}$ (172)

Using the primitive functions of the two integrals of the right hand term, $-½x_1{}^2$ and $\frac{1}{2}\ln(x_1{}^2+(z_2-z_1)^2)$ respectively, Eq.(170) may now be rewritten:

$$-\frac{1}{2}J^2 \int_{z_2=0}^{t} dz_2 \int_{z_1=0}^{t} dz_1 (\frac{1}{2}w^2 \ln(w^2+(z_2-z_1)^2) - \frac{1}{2}w^2 + \frac{1}{2}(z_2-z_1)^2 \ln(w^2+(z_2-z_1)^2) -$$

$$-\frac{1}{2}(z_2-z_1)^2\ln((z_2-z_1)^2))) \qquad (173)$$

Thereafter the first term of Eq. (163) will be treated.

However, integrating a function of x_1 between 0 and w is equal to integrating the same function with argument $w-x_1$ along that same interval, as is being done in Eq.(164).

Hence,

$\vec{F}''(firstterm) = \vec{F}'(\sec ondterm)$ (174) and their sum will be

$$\vec{F}''(firstterm)+\vec{F}''(\sec ondterm)=$$

$$-\frac{1}{2}J^2\int_{z_2=0}^{t}dz_2\int_{z_1=0}^{t}dz_1(w^2\ln(w^2+(z_2-z_1)^2)-w^2+(z_2-z_1)^2\ln(w^2+(z_2-z_1)^2)-$$

$$-(z_2-z_1)^2\ln((z_2-z_1)^2))) \qquad (175)$$

The third term of Eq. (169) will be very easily treated.

$$\vec{F}''(thirdterm)=-\frac{1}{2}J^2\int_{z_2=0}^{t}dz_2\int_{z_1=0}^{t}dz_1\int_{x_1=0}^{w}dx_1(-2w)=-\frac{1}{2}J^2\int_{z_2=0}^{t}dz_2\int_{z_1=0}^{t}dz_1(-2w^2) \qquad (176)$$

The two last terms (fourth and fifth) require the usage of the following integration formula:

$$\int \arctan z dz = z\arctan z - ½\ln(1+z^2) \quad [15] \qquad (177)$$

$$\vec{F}''(fifthterm)=-\frac{1}{2}J^2\int_{z_2=0}^{t}dz_2\int_{z_1=0}^{t}dz_1\int_{x_1=0}^{w}2dx_1(z_2-z_1)\arctan\frac{x_1}{z_2-z_1} \qquad (178)$$

In order to use Eq. (171) it is obviously needed a variable substitution $x=\frac{x_1}{z_2-z_1}$ (179)

This leads to:

$$\vec{F}''(fifthterm)=-\frac{1}{2}J^2\int_{z_2=0}^{t}dz_2\int_{z_1=0}^{t}dz_1\int_{x=0}^{w/(z_2-z_1)}2dx(z_2-z_1)^2\arctan x \qquad (180)$$

which develops to

$$-\frac{1}{2}J^2\int_{z_2=0}^{t}dz_2\int_{z_1=0}^{t}dz_1(2(z_2-z_1)^2\left(x\arctan x-½\ln(1+x^2)\right)_{x=0}^{w/(z_2-z_1)}) \qquad (181)$$

and, finally,

$$-\frac{1}{2}J^2\int_{z_2=0}^{t}dz_2\int_{z_1=0}^{t}dz_1(2(z_2-z_1)w\arctan\frac{w}{z_2-z_1}-(z_2-z_1)^2\ln(1+(\frac{w}{z_2-z_1})^2)) \qquad (182)$$

However, integrating a function of x_1 between 0 and w is equal to integrating the same function with argument $w-x_2$ along the same interval, as is being done in Eq.(182).

Hence,

$$\vec{F}''(fourthterm)=\vec{F}''(fifthterm)=$$

$$-\frac{1}{2}J^2\int_{z_2=0}^{t}dz_2\int_{z_1=0}^{t}dz_1(2(z_2-z_1)w\arctan\frac{w}{z_2-z_1}-(z_2-z_1)^2\ln(1+(\frac{w}{z_2-z_1})^2)) \qquad (183)$$

$$\vec{F}''(fourthterm)+\vec{F}''(fifthterm)=$$

$$-\frac{1}{2}J^2\int_{z_2=0}^{t}dz_2\int_{z_1=0}^{t}dz_1(4(z_2-z_1)w\arctan\frac{w}{z_2-z_1}-2(z_2-z_1)^2\ln(1+(\frac{w}{z_2-z_1})^2)) \qquad (184)$$

Making now the sum of all terms, due to Eq. (175), (176) and (184) it is now possible to write

$$\vec{F}'' = -\tfrac{1}{2}J^2 \int_{z_2=0}^{t} dz_2 \int_{z_1=0}^{t} dz_1 (w^2 \ln(w^2 + (z_2 - z_1)^2) - 3w^2 + (z_2 - z_1)^2 \ln(w^2 + (z_2 - z_1)^2). -$$

$$-(z_2 - z_1)^2 \ln((z_2 - z_1)^2) + 4(z_2 - z_1) w \arctan \frac{w}{z_2 - z_1} - 2(z_2 - z_1)^2 \ln(1 + (\frac{w}{z_2 - z_1})^2))$$

(185)

This integral will now be solved for one term after another.
Before doing so, it would be practical to first simplify the integrand and put its respective terms on one row each, as will be done in the following treatment.

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 \qquad (186)$$

$$\vec{F}_1 = -\tfrac{1}{2}J^2 \int_{z_2=0}^{t} dz_2 \int_{z_1=0}^{t} dz_1 (4(z_2 - z_1) w \arctan \frac{w}{z_2 - z_1}) \qquad (187)$$

$$\vec{F}_2 = -\tfrac{1}{2}J^2 \int_{z_2=0}^{t} dz_2 \int_{z_1=0}^{t} dz_1 (-(z_2 - z_1)^2 \ln(w^2 + (z_2 - z_1)^2)) \qquad (188)$$

$$\vec{F}_3 = -\tfrac{1}{2}J^2 \int_{z_2=0}^{t} dz_2 \int_{z_1=0}^{t} dz_1 ((z_2 - z_1)^2 \ln(z_2 - z_1)^2) \qquad (189)$$

$$\vec{F}_4 = -\tfrac{1}{2}J^2 \int_{z_2=0}^{t} dz_2 \int_{z_1=0}^{t} dz_1 \; w^2 \ln(w^2 + (z_2 - z_1)^2) \qquad (190)$$

$$\vec{F}_5 = -\tfrac{1}{2}J^2 \int_{z_2=0}^{t} dz_2 \int_{z_1=0}^{t} dz_1 (-3w^2) \qquad (191)$$

/

Solving $\vec{F}_1$ (187), the final result given in expression(227)

Beginning thus with the first integral, $\vec{F}_1$
First one has to observe a property of the integrand that makes it impossible to perform the integration straightforwardly. The denominator namely becomes zero at the point where $z_1 = z_2$. Since the integrand is an arctan function, its value jumps π at that point. In order to succeed with the integral, one has to divide the integration into two procedures, one below that singular point and one above it. Hence,

$$\vec{F}_1 = -\tfrac{1}{2}J^2 \int_{z_2=0}^{t} dz_2 \int_{z_1=0}^{z_2} dz_1 (4(z_2 - z_1) w \arctan \frac{w}{z_2 - z_1})$$
$$-\tfrac{1}{2}J^2 \int_{z_2=0}^{t} dz_2 \int_{z_1=z_2}^{t} dz_1 (4(z_2 - z_1) w \arctan \frac{w}{z_2 - z_1}) \qquad (192)$$

Using the variable substitution $x = \dfrac{w}{z_2 - z_1}$ (193) (x being used as a local variable as usual)
it will be possible to use integral (177)
which leads to $\vec{F}_1 =$

$$-\tfrac{1}{2}J^2\int_{z_1=0}^{t}dz_1\int_{x=-w/z_1}^{-\infty}dx(-4(\frac{w}{x})^3\arctan x)dx \;-\tfrac{1}{2}J^2\int_{z_1=0}^{t}dz_1\int_{x\to+\infty}^{x=t}dx(-4(\frac{w}{x})^3\arctan x)dx =$$

$$-\tfrac{1}{2}J^2(-4w^3)\int_{z_1=0}^{t}dz_1\int_{x=-w/z_1}^{-\infty}(\frac{1}{x^3}\arctan x)dx \quad -\tfrac{1}{2}J^2(-4w^3)\int_{z_1=0}^{t}dz_1\int_{x\to+\infty}^{x=t}(\frac{1}{x^3}\arctan x)dx$$

(194)

The next step to be performed is to solve the last integral. That will be done by using both the partial integral formula (11) and Eq.(177).

$$\int_{x=-w/z_1}^{w/(t-z_1)}(\frac{1}{x^3}\arctan x)dx = \frac{1}{x^3}(x\arctan x-\tfrac{1}{2}\ln(1+x^2))-\int(-\frac{3}{x^4})(x\arctan x-\tfrac{1}{2}\ln(1+x^2)dx$$

(195)

Now it is found that two of the terms are the same kind of functions. Hence, one may simplify Eq. (195) into

$$\int((-\frac{2}{x^3})\arctan x)dx = \frac{1}{x^2}\arctan x-\frac{1}{2x^3}\ln(1+x^2)-\tfrac{1}{2}\int\frac{3}{x^4}\ln(1+x^2)dx \qquad (196)$$

The last term of that expression may now be solved using partial integration (11), which leads to:

$$\int\frac{1}{x^4}\ln(1+x^2)dx = -\frac{1}{3x^3}\ln(1+x^2)-\int(-\frac{1}{3x^3})\frac{2x}{1+x^2}dx = -\frac{1}{3x^3}\ln(1+x^2)+\frac{2}{3}\int\frac{1}{x^2}\frac{1}{1+x^2}dx$$

(197)

The last term may be solved by separating into partial fractions:

$$\frac{1}{x^2}\frac{1}{1+x^2}=\frac{1}{x^2}-\frac{1}{1+x^2} \qquad (198)$$

Hence, Eq. (197) develops to:

$$\int\frac{1}{x^4}\ln(1+x^2)dx = -\frac{1}{3x^3}\ln(1+x^2)-\int(-\frac{1}{3x^3})\frac{2x}{1+x^2}dx = -\frac{1}{3x^3}\ln(1+x^2)+\frac{2}{3}\int\frac{dx}{x^2}-\frac{2}{3}\int\frac{dx}{1+x^2}$$

(199)

Using now Eq. (199) into Eq. (196) gives accordingly:

$$\int((-\frac{2}{x^3})\arctan x)dx = \frac{1}{x^2}\arctan x+\frac{1}{x}+\arctan x$$

(200)

But going back to Eq. (194) leads now to

$$d^2\vec{F}_1 = -\tfrac{1}{2}J^2(2w^3)\int_{z_1=0}^{t}dz_1\,((1/x^2)\arctan x+1/x+\arctan x)_{x=-w/z_1}^{-\infty}$$

$$--\tfrac{1}{2}J^2(2w^3)\int_{z_1=0}^{t}dz_1\,((1/x^2)\arctan x+1/x+\arctan x)_{x\to+\infty_1}^{x=w/(t-z_1)} \qquad (201)$$

Evaluating this expression now gives

$$\vec{F}_1 = -\tfrac{1}{2}J^2(2w^3)\int_{z_1=0}^{t}dz_1((\frac{t-z_1}{w})^2\arctan\frac{w}{t-z_1}+\frac{t-z_1}{w}+\arctan\frac{w}{t-z_1}-(\frac{z_1}{w})^2\arctan\frac{w}{z_1}+\frac{z_1}{w}+\arctan\frac{w}{z_1}-\pi)$$

(202)

This integral will now be solved for one term after another.

Before doing so, it would be practical to first simplify the integrand and put its respective terms on one row each, as will be done in the following treatment.

$$\bar{F}_{11a} = -\tfrac{1}{2}J^2(2w^3)\int_{z_1=0}^{t} dz_1 \left(\frac{t-z_1}{w}\right)^2 \arctan\frac{w}{t-z_1} \quad (203)$$

$$\bar{F}_{11b} = -\tfrac{1}{2}J^2(2w^3)\int_{z_1=0}^{t} dz_1 \left(\frac{z_1}{w}\right)^2 \arctan\frac{w}{z_1} \quad (204)$$

$$\bar{F}_{12a} = -\tfrac{1}{2}J^2(2w^3)\int_{z_1=0}^{t} dz_1 \arctan\frac{w}{t-z_1} \quad (205)$$

$$\bar{F}_{12b} = -\tfrac{1}{2}J^2(2w^3)\int_{z_1=0}^{t} dz_1 \arctan\frac{w}{z_1} \quad (206)$$

$$\bar{F}_{13} = -\tfrac{1}{2}J^2(2w^3)\int_{z_1=0}^{t} dz_1 \frac{t}{w} \quad (207)$$

$$\bar{F}_{14} = -\tfrac{1}{2}J^2(2w^3)\int_{z_1=0}^{t} dz_1 (-\pi) \quad (208)$$

In the last case the two linear terms (3rd and 6th in Eq. (202)) have already been combined to one.

In order to solve (203) the following variable substitution will be done: $\frac{w}{t-z_1} = x$. This makes the boarders change: $z_1 = 0$ is transformed to $x = \frac{w}{t}$ and $z_1 = t$ to $x \to \infty$ .Further one can rewrite $dz_1 = \frac{w}{x^2}dx$. Hence, one may now rewrite Eq. (203):

$$\bar{F}_{11a} = -\tfrac{1}{2}J^2(2w^3)\int_{x=w/t}^{\infty} dx \frac{1}{x^2}(\arctan x)\frac{w}{x^2} = -\tfrac{1}{2}J^2(2w^4)\int dx \frac{1}{x^4}\arctan x \quad (209)$$

Using the partial integration formula (11), the last integral of (209) may be rewritten:

$$\int dx \frac{1}{x^4}\arctan x = -\frac{1}{3x^3}\arctan x - \int dx\left(-\frac{1}{3x^3}\right)\frac{1}{1+x^2} \quad (210)$$

In order to proceed, it must first be realized that

$$\frac{1}{x^3}\frac{1}{1+x^2} = -\frac{1}{x} + \frac{1}{x^3} + \frac{x}{1+x^2} \quad (211)$$

Inserting all these partial results into Eq. (209) gives accordingly:

$$\bar{F}_{11a} = -\tfrac{1}{2}J^2(2w^4)\left((-1/3x^3)\arctan x\right)_{x=w/t}^{\infty} + \frac{1}{3}\left(-\int_{x=w/t}^{\infty} dx \frac{1}{x} + \int_{x=w/t}^{\infty} dx \frac{1}{x^3} + \int dx \frac{x}{1+x^2}\right)$$

(212)

However, this expression may be further simplified after performing the integrals:

$$\bar{F}_{11a} = -\tfrac{1}{2}J^2(2w^4)\left((-1/3x^3)\arctan x - (1/6x^2) + (1/6)\ln((1+x^2)/x^2)\right)_{x=w/t}^{\infty}$$

(213)

Inserting the boarder values of the primitive function now gives accordingly:

$$\bar{F}_{11a} = -\tfrac{1}{2}J^2(2w^4)\left(\frac{1}{3}\left(\frac{t}{w}\right)^3 \arctan\frac{w}{t} + \frac{1}{6}\left(\frac{t}{w}\right)^2 - \frac{1}{6}\ln\frac{t^2+w^2}{w^2}\right) \quad (214)$$

It remains now to solve two integrals in order to finally evaluate Eq. (189). In order to proceed, partial integration according to Eq. (12) will favourably be used.

Eq. (204) reminds of Eq. (203).It may be recalled that integrating a function of z_1 from $0 \to t$ is equal to integrating the same function, but with argument $t - z_1$ from $0 \to t$.

Hence, $\bar{F}_{11b} = \bar{F}_{11a} = -½J^2(2w^4)(\frac{1}{3}(\frac{t}{w})^3 \arctan\frac{w}{t} + \frac{1}{6}(\frac{t}{w})^2 - \frac{1}{6}\ln\frac{t^2+w^2}{w^2})$ (215)

For the reader's convenience, however, this is also done here:

In order to solve (204) the following variable substitution will be done: $\frac{w}{z_1} = x$. This makes the boarders change: $z_1 = 0$ is transformed to $x \to \infty$ and $z_1 = t$ to $x = \frac{w}{t}$. .Further one can rewrite $dz_1 = -\frac{w}{x^2}dx$. Hence, one may now rewrite Eq. (204):

$$\bar{F}_{11b} = -½J^2(2w^3)\int_{'x\to\infty}^{w/t} dx\frac{1}{x^2}(\arctan x)(-\frac{w}{x^2}) = -½J^2(-2w^4)\int_{x\to\infty}^{w/t} dx\frac{1}{x^4}\arctan x \quad (216)$$

and, hence, $\bar{F}_{11b} = \bar{F}_{11a}$ (217)

When now dealing with the following integral, Eq. (205), the same variable substitution as in Eq. (203) has to be done. This leads to:

$$\bar{F}_{12a} = -½J^2(2w^3)\int_{z_1=0}^{t} dx\frac{w}{x^2}\arctan x = -½J^2(2w^4)\int_{z_1=0}^{t} dx\frac{1}{x^2}\arctan x \quad (218)$$

Using the partial integration formula (11), the last integral of (218) may be rewritten

$$\int_{z_1=0}^{t} dx\frac{1}{x^2}\arctan x = -\frac{1}{x}\arctan x - \int dx(-\frac{1}{x})\frac{1}{1+x^2} \quad (219)$$

In order to proceed, it must first be realized that

$$\frac{1}{x}\frac{1}{1+x^2} = \frac{1}{x} - \frac{x}{1+x^2} \quad (220)$$

Inserting all these partial results into Eq. (209) gives accordingly:

$$\bar{F}_{12a} = -½J^2(2w^4)(\left((-1/x)\arctan x\right)_{x=w/t}^{\infty} + \int_{x=w/t}^{\infty} dx\frac{1}{x} - \int_{x=w/t}^{\infty} dx\frac{x}{1+x^2}) \quad (221)$$

However, this expression may be further simplified after performing the integrals:

$$\bar{F}_{12a} = -½J^2(2w^4)\left((-1/x)\arctan x - (1/2)\ln((1+x^2)/x^2)\right)_{x=w/t}^{\infty} \quad (222)$$

Inserting the boarder values of the primitive function now gives accordingly:

$$\bar{F}_{12aa} = -½J^2(2w^4)((\frac{t}{w})\arctan\frac{w}{t} + \frac{1}{2}\ln\frac{t^2+w^2}{w^2}) \quad (223)$$

Eq. (206) reminds of Eq. (205).It may be recalled that integrating a function of z_1 from $0 \to t$ is equal to integrating the same function, but with argument $t - z_1$ from $0 \to t$.

Hence, $d^2\bar{F}_{12b} = d^2\bar{F}_{12aa} = -½J^2(2w^4)((\frac{t}{w})\arctan\frac{w}{t} + \frac{1}{2}\ln\frac{t^2+w^2}{w^2})$ (224)

The result will, however, this time not be derived especially.
The final integral belonging to Eq. (202), Eq. (207), is very easily performed, since the integrand is only a constant. Hence,
$\vec{F}_{13} = -½J(2w^2t^2)$ (225)

In order to simplify the reader's overview, all the results of integrating Eq. (202) will here be gathered:
The result is:
$$\vec{F}_1 = -½J^2((\frac{4wt^3}{3} + 4w^3t)\arctan\frac{w}{t} + \frac{2w^2t^2}{3} + 2w^2t^2 + (2w^4 - \frac{2w^4}{3})\ln(t^2+w^2) + (\frac{2w^4}{3} - 2w^4)\ln w^2 - 2w^3t\pi)$$

(226)
This result may be further simplified to

$$\vec{F}_1 = -½J^2((\frac{4w^3t}{3} + 4w^3t)\arctan\frac{w}{t} + \frac{8w^2t^2}{3} + 2w^2t^2 + \frac{4w^4}{3}\ln(t^2+w^2) - \frac{4w^4}{3}\ln w^2 - 2w^3t\pi)$$
(227)
However, the two arctan terms will be kept separated from each others, due to needs that will appear later.

Solving $\vec{F}_2$ (188), the final result given in expression(263)
Beginning thus with the second integral, $\vec{F}_2$, closer defined in Eq. (188)

Using the variable substitution $x = z_2 - z_1$ (228) (x being used as a local variable as usual)
This makes the boarders change: $z_2 = 0$ is transformed to $x = -z_1$ and $z_2 = t$ to $x = t - z_1$ .Further one can rewrite $dz_2 = dx$. Hence, one may now rewrite Eq. (188):
$$\vec{F}_2 = -½J^2\int_{z_1=0}^{t}dz_1\int_{x=-z_1}^{t-z_1}dx(-x^2\ln(w^2+x^2)) \qquad (229)$$
In order to solve this integral, the partial integral formula (11) will be used:
$$\int dx(x^2\ln(w^2+x^2)) = \frac{x^3}{3}\ln(w^2+x^2) - \int dx\frac{x^3}{3}\frac{2x}{w^2+x^2} \qquad (230)$$
The last term of Eq. (230) may be simplified according to the following steps:

$$\frac{2}{3}\int dx\frac{x^4}{w^2+x^2} = \frac{2}{3}\int dx\frac{x^2(x^2+w^2-w^2)}{w^2+x^2} = \frac{2}{3}\int dx(x^2) - \frac{2}{3}w^2\int dx\frac{x^2}{w^2+x^2} \qquad (231)$$
Now $\frac{2}{3}w^2\int dx\frac{x^2}{w^2+x^2} = \frac{2}{3}w^2\int dx\frac{x^2+w^2-w^2}{w^2+x^2} = \frac{2}{3}w^2\int dx - \frac{2}{3}w^4\int dx\frac{1}{w^2+x^2}$ (232)
and $\int dx\frac{1}{w^2+x^2} = \frac{1}{w}\arctan\frac{x}{w}$ (233) [6]
and, hence, using all these results, Eq. (229) may be rewritten:
$$\vec{F}_2 = -½J^2\int_{z_1=0}^{t}dz_1(-(x^3/3)\ln(w^2+x^2) + (2x^3/9) - (2/3)w^2x)_{x=-z_1}^{t-z_1})$$

$$-\tfrac{1}{2}J^2\int_{z_1=0}^{t} dz_1\left((2/3)w^3\arctan(x/w)\Big|_{x=-z_1}^{t-z_1}\right)$$

(234)

Inserting the values of the primitive function above gives

$$\bar{F}_2=-\tfrac{1}{2}J^2\int_{z_1=0}^{t} dz_1\left(-\frac{(t-z_1)^3}{3}\ln(w^2+(t-z_1)^2)+\frac{2(t-z_1)^3}{9}-\frac{2}{3}w^2(t-z_1)+\frac{2}{3}w^3\arctan\frac{t-z_1}{w}\right)$$
$$-\tfrac{1}{2}J^2\int_{z_1=0}^{t} dz_1\left(\left(\frac{(-z_1)^3}{3}\right)\ln(w^2+(-z_1)^2)-\frac{2(-z_1)^3}{9}+\frac{2}{3}w^2(-z_1)-\frac{2}{3}w^3\arctan\frac{-z_1}{w}\right)$$

(235)

This integral will now be solved for one term after another.
Before doing so, it would be practical to first simplify the integrand and put its respective terms on one row each, as will be done in the following treatment.

$$\bar{F}_{21a}=-\tfrac{1}{2}J^2\int_{z_1=0}^{t} dz_1\left(-\frac{(t-z_1)^3}{3}\ln(w^2+(t-z_1)^2)\right) \qquad (236)$$

$$\bar{F}_{21b}=-\tfrac{1}{2}J^2\int_{z_1=0}^{t} dz_1\left(\left(\frac{(-z_1)^3}{3}\right)\ln(w^2+(-z_1)^2)\right) \qquad (237)$$

$$\bar{F}_{22a}=-\tfrac{1}{2}J^2\int_{z_1=0}^{t} dz_1\left(\frac{2}{3}w^3\arctan\frac{t-z_1}{w}\right) \qquad (238)$$

$$\bar{F}_{22b}=-\tfrac{1}{2}J^2\int_{z_1=0}^{t} dz_1\left(-\frac{2}{3}w^3\arctan\frac{-z_1}{w}\right) \qquad (239)$$

$$\bar{F}_{23a}=-\tfrac{1}{2}J^2\int_{z_1=0}^{t} dz_1\,\frac{2(t-z_1)^3}{9} \qquad (240)$$

$$\bar{F}_{23b}=-\tfrac{1}{2}J^2\int_{z_1=0}^{t} dz_1\left(-\frac{2(-z_1)^3}{9}\right) \qquad (241)$$

$$\bar{F}_{24a}=-\tfrac{1}{2}J^2\int_{z_1=0}^{t} dz_1\left(-\frac{2}{3}w^2(t-z_1)\right) \qquad (242)$$

$$\bar{F}_{24b}=-\tfrac{1}{2}J^2\int_{z_1=0}^{t} dz_1\left(\frac{2}{3}w^2(-z_1)\right) \qquad (243)$$

In order to solve Eq. (236) the following variable substitution will be done: $t-z_1=x$. This makes the boarders change: $z_1=0$ is transformed to $x=t$ and $z_1=t$ to $x=0$ .Further one can rewrite $dz_1=-dx$. Hence, one may now rewrite Eq. (236):

$$\bar{F}_{21a}=-\tfrac{1}{2}J^2\int_{z_1=0}^{t}(-dx)\left(-\frac{x^3}{3}\ln(w^2+x^2)\right)=-\tfrac{1}{2}J^2\left(\frac{1}{3}\right)\int_{z_1=0}^{t} dx(x^3\ln(w^2+x^2)) \qquad (244)$$

Using now the partial integration formula (11), the last integral of (244) may be rewritten:

$$\int_{z_1=0}^{t} dx(x^3 \ln(w^2+x^2)) = \frac{x^4}{4}\ln(x^2+w^2) - \frac{1}{2}\int dx \frac{x^5}{x^2+w^2} \qquad (245)$$

Now $-\frac{1}{2}\int dx \frac{x^5}{x^2+w^2} = -\frac{1}{2}\int dx \frac{x^3(x^2+w^2-w^2)}{x^2+w^2} = -\frac{1}{2}\int dx(x^3) + \frac{w^2}{2}\int dx \frac{x^3}{x^2+w^2}$ (246)

and $\frac{w^2}{2}\int dx \frac{x^3}{x^2+w^2} = \frac{w^2}{2}\int dx \frac{x(x^2+w^2-w^2)}{x^2+w^2} = \frac{w^2}{2}\int dx(x) - \frac{w^4}{2}\int dx \frac{x}{x^2+w^2}$ (247)

and further $-\frac{w^4}{2}\int dx \frac{x}{x^2+w^2} = -\frac{w^4}{4}\ln(x^2+w^2)$ (248)

These results make it now possible to rewrite Eq. (244) according to:

$$\vec{F}_{21a} = -½J^2(\frac{1}{3})((x^4/4)\ln(x^2+w^2) - (x^4/8) + (w^2x^2)/4 - (w^4/4)\ln(x^2+w^2))_{x=t}^{0}$$

$$= -½J^2(-\frac{w^4}{12}\ln w^2 - \frac{t^4}{12}\ln(t^2+w^2) + \frac{t^4}{24} - \frac{w^2t^2}{12} + \frac{w^4}{12}\ln(t^2+w^2)) \qquad (249)$$

This expression can be further simplified:

$$\vec{F}_{21a} = -½J^2(-\frac{w^4}{12}\ln w^2 + (\frac{w^4}{12} - \frac{t^4}{12})\ln(t^2+w^2) + \frac{t^4}{24} - \frac{w^2t^2}{12}) \qquad (250)$$

Eq. (237) reminds of Eq. (236).It may be recalled that integrating a function of z_1 from $0 \to t$ is equal to integrating the same function, but with argument $t - z_1$ from $0 \to t$.

Hence, $\vec{F}_{21b} = \vec{F}_{21a} = -½J^2(-\frac{w^4}{12}\ln w^2 + (\frac{w^4}{12} - \frac{t^4}{12})\ln(t^2+w^2) + \frac{t^4}{24} - \frac{w^2t^2}{12})$ (251)

The result will, however, this time not be derived separately.

In order to solve Eq. (238) the following variable substitution will be done: $\frac{t-z_1}{w} = x$. This makes the boarders change: $z_1 = 0$ is transformed to $x = \frac{t}{w}$ and $z_1 = t$ to $x = 0$ .Further one can rewrite $dz_1 = -wdx$. Hence, one may now rewrite Eq. (238):

$$\vec{F}_{22a} = -½J^2(-\frac{2w^4}{3})\int_{x=t/w}^{0} dx(\arctan x) \qquad (252)$$

Now $\int dx \arctan x = x\arctan x - ½\ln(1+x^2)$ (177) [15]

Hence, Eq. (252) develops to

$$\vec{F}_{22a} = -½J^2(-\frac{2w^4}{3})(((2w^4/3)(x\arctan x - (1/2)\ln(1+x^2))_{x=t/w}^{0}) \qquad (253)$$

Inserting the values of x into the primitive function gives

$$\vec{F}_{22a} = -½J^2(\frac{2w^4}{3})(\frac{t}{w}\arctan\frac{t}{w} - \frac{1}{2}\ln(t^2+w^2) + \frac{1}{2}\ln w^2) \qquad (254)$$

Eq. (239) reminds of Eq. (238).It may be recalled that integrating a function of z_1 from $0 \to t$ is equal to integrating the same function, but with argument $t - z_1$ from $0 \to t$.

Hence, $\vec{F}_{22b} = \vec{F}_{22a} = -½J^2(\frac{2w^4}{3})(\frac{t}{w}\arctan\frac{t}{w} - \frac{1}{2}\ln(t^2+w^2) + \frac{1}{2}\ln w^2)$ (255)

The result will, however, this time not be derived separately.

Eq. (240) is solved rather straightforwardly. First a variable substitution has to be made: $t - z_1 = x$. This makes the boarders change: $z_1 = 0$ is transformed to $x = t$ and $z_1 = t$ to $x = 0$ .Further one can rewrite $dz_1 = -dx$. Hence, one may now rewrite Eq. (240):

$$\vec{F}_{23a} = -½J^2 \int_{x=t}^{0} (-dx)\frac{2x^3}{9} \qquad (256)$$

giving thus the result $d^2\vec{F}_{23a} = -½J^2(\frac{t^4}{18})$ (257)

Eq. (241) reminds of Eq. (240).It may be recalled that integrating a function of z_1 from $0 \to t$ is equal to integrating the same function, but with argument $t - z_1$ from $0 \to t$.

Hence, $\vec{F}_{23b} = \vec{F}_{23a} = -½J^2(\frac{t^4}{18})$ (258)

The result will, however, this time not be derived separately.

Eq. (242) will be solved rather straightforwardly. First a variable substitution has to be made: $t - z_1 = x$. This makes the boarders change: $z_1 = 0$ is transformed to $x = t$ and $z_1 = t$ to $x = 0$ .Further one can rewrite $dz_1 = -dx$. Hence, one may now rewrite Eq. (221):

$$\vec{F}_{24a} = -½J^2(\frac{-2w^2}{3}) \int_{z_1=0}^{t} (-dx)(x) \qquad (259)$$

giving easily the result $\vec{F}_{24a} = -½J^2(-\frac{w^2t^2}{3})$ (260)

Eq. (243) reminds of Eq. (242).It may be recalled that integrating a function of z_1 from $0 \to t$ is equal to integrating the same function, but with argument $t - z_1$ from $0 \to t$.

Hence, $\vec{F}_{24b} = \vec{F}_{24a} = -½J^2(-\frac{w^2t^2}{3})$ (261)

In order to simplify the reader's overview, all the results of integrating the integral equations (236) to (244) will here be gathered:

The result is:

$$\vec{F}_2 = -½J^2(\frac{4w^3t}{3}\arctan\frac{t}{w} - \frac{w^2t^2}{6} - \frac{2w^2t^2}{3} + \frac{t^4}{9} + \frac{t^4}{12} + (\frac{w^4}{6} - \frac{t^4}{6} - \frac{2w^4}{3})\ln(t^2 + w^2) + (\frac{2w^4}{3} - \frac{w^4}{6})\ln w^2) \qquad (262)$$

which may be further simplified according to

$$\vec{F}_2 = -½J^2(\frac{4w^3t}{3}\arctan\frac{t}{w} - \frac{5w^2t^2}{6} + \frac{7t^4}{36} + (\frac{-w^4}{2} - \frac{t^4}{6})\ln(t^2 + w^2) + \frac{w^4}{2}\ln w^2) \qquad (263)$$

Solving $\vec{F}_3$ (189), the final result given in expression(284)

Beginning thus with the third integral, $\vec{F}_3$

Using the variable substitution $x = z_2 - z_1$ (264) (x being used as a local variable as usual)

This makes the boarders change: $z_2 = 0$ is transformed to $x = -z_1$ and $z_2 = t$ to $x = t - z_1$ .Further one can rewrite $dz_2 = dx$. Hence, one may now rewrite Eq. (189):

$$\vec{F}_3 = -\tfrac{1}{2} J^2 \int_{z1=0}^{t} dz_1 \int_{x=-z_1}^{t-z_1} dx (x^2 \ln x^2) \qquad (265)$$

In order to solve the integral, the partial integral formula (11) will be used:

$$\int dx (x^2 \ln x^2) = \frac{x^3}{3} \ln x^2 - \int dx (\frac{x^3}{3}) \frac{2x}{x^2} = \frac{x^3}{3} \ln x^2 - \frac{2x^3}{9} \qquad (266)$$

Solving the integral for the given interval gives

$$\vec{F}_3 = -\tfrac{1}{2} J^2 \int_{z1=0}^{t} dz_1 (((x^3/3) \ln x^2 - (2/9) x^3)_{x=-z_1}^{t-z_1}) \qquad (267)$$

Inserting the values of x into the primitive function gives

$$\vec{F}_3 = -\tfrac{1}{2} J^2 \int_{z1=0}^{t} dz_1 (\frac{(t-z_1)^3}{3} \ln (t-z_1)^2 - \frac{2}{9} (t-z_1)^3 - \frac{(-z_1)^3}{3} \ln (-z_1)^2 + \frac{2}{9} (-z_1)^3) \qquad (268)$$

simpler written $d^2 \vec{F}_3 = -\tfrac{1}{2} J^2 \int_{z1=0}^{t} dz_1 (\frac{(t-z_1)^3}{3} \ln (t-z_1)^2 - \frac{2}{9} (t-z_1)^3 + \frac{z_1^{\ 3}}{3} \ln z_1^{\ 2} - \frac{2}{9} z_1^{\ 3})$

(269)

This integral will now be solved for one term after another.
Before doing so, it would be practical to first simplify the integrand and put its respective terms on one row each, as will be done in the following treatment.

$$\vec{F}_{31a} = -\tfrac{1}{2} J^2 \int_{z1=0}^{t} dz_1 \frac{(t-z_1)^3}{3} \ln (t-z_1)^2 \qquad (270)$$

$$\vec{F}_{31b} = -\tfrac{1}{2} J^2 \int_{z1=0}^{t} dz_1 \frac{z_1^{\ 3}}{3} \ln z_1^{\ 2} \qquad (271)$$

$$\vec{F}_{32a} = -\tfrac{1}{2} J^2 \int_{z1=0}^{t} dz_1 (-\frac{2}{9} (t-z_1)^3) \qquad (272)$$

$$\vec{F}_{32b} = -\tfrac{1}{2} J^2 \int_{z1=0}^{t} dz_1 (-\frac{2}{9} z_1^{\ 3}) \qquad (273)$$

In order to solve Eq. (270) the following variable substitution will be done: $t - z_1 = x$. This makes the boarders change: $z_1 = 0$ is transformed to $x = t$ and $z_1 = t$ to $x = 0$ .Further one can rewrite $dz_1 = -dx$. Hence, one may now rewrite Eq. (270)

:

$$\vec{F}_{31a} = -\tfrac{1}{2} J^2 \int_{z1=0}^{t} (-dx) \frac{x^3}{3} \ln x^2 \qquad (274)$$

Using now the partial integration formula (11), the last integral of Eq. (274) may be rewritten:

$$\int_{z1=0}^{t} dx \frac{x^3}{3}\ln x^2 = \frac{x^4}{12}\ln x^2 - \int dx \frac{x^4}{12}\frac{2x}{x^2} = \frac{x^4}{12}\ln x^2 - \frac{x^4}{24} \qquad (275)$$

and hence, Eq. (253) can be written:

$$\vec{F}_{31a} = -½J^2(-1)(((x^4/12)\ln x^2 - (1/24)x^4)_{x=t}^{0}) \qquad (276)$$

One term therein may cause some confusion, namely::

$$\lim_{x\to 0}(\frac{x^4}{12})\ln x^2 \qquad (277)$$

However, since the term $\propto x^4$ decreases more rapidly than $\ln x^2$ increases, as $x \to 0$ this term (277) as a whole, approaches zero.

This can easily be proven mathematically, by comparing $\lim_{x\to 0} e^{(\frac{x^4}{12})\ln x^2}$ with $\lim_{x\to 0}(\frac{x^4}{12})\ln x^2$, the former always larger than the latter and the former approaching zero as x goes to zero. Hence,

$$\vec{F}_{31a} = -½J^2(\frac{t^4}{24}\ln t^2 - \frac{t^4}{24}) \qquad (278)$$

Eq. (271) reminds of Eq. (270).It may be recalled that integrating a function of z_1 from $0 \to t$ is equal to integrating the same function, but with argument $t - z_1$ from $0 \to t$.

Hence, $\vec{F}_{31b} = \vec{F}_{31a} = -½J^2(\frac{t^4}{24}\ln t^2 - \frac{t^4}{24})$ (279)

The result will, however, this time not be derived separately.

In order to solve Eq. (272) the same variable substitution as in the preceding case may be used. But apparently, the solution can immediately be realized due to the simplicity of the expression, and accordingly,

$$\vec{F}_{32a} = -½J^2(\frac{2}{9}((1/4)(t-z_1)^4)_{z_1=0}^{t}) \qquad (280)$$

Inserting the values of z_1 into the primitive function gives

$$\vec{F}_{32a} = -½J^2(\frac{-t^4}{18}) \qquad (281)$$

Eq. (273) reminds of Eq. (272).It may be recalled that integrating a function of z_1 from $0 \to t$ is equal to integrating the same function, but with argument $t - z_1$ from $0 \to t$.
Hence,

$$\vec{F}_{32b} = \vec{F}_{32a} = -½J^2(\frac{-t^4}{18}) \qquad (282)$$

In order to simplify the reader's overview, all the results of integrating the integral equations (270) to (273) will here be gathered:

$$\vec{F}_3 = -½J^2(\frac{t^4}{6}\ln t^2 - \frac{t^4}{12} - \frac{t^4}{9}) \qquad (283)$$

or even simpler:

$$\vec{F}_3 = -½J^2(\frac{t^4}{6}\ln t^2 - \frac{7t^4}{36}) \qquad (284)$$

Solving $\vec{F}_4$ (190), the final result given in expression(313)

Beginning thus with the third integral, $\vec{F}_4$
Using the variable substitution $x = z_2 - z_1$ (285) (x being used as a local variable as usual)
This makes the boarders change: $z_2 = 0$ is transformed to $x = -z_1$ and $z_2 = t$ to $x = t - z_1$
.Further one can rewrite $dz_2 = dx$. Hence, one may now rewrite Eq. (190):

$$\vec{F}_4 = -½J^2(w^2)\int_{z_1=0}^{t} dz_1 \int_{x=-z_1}^{t-z_1} dx \ln(x^2 + w^2) \qquad (286)$$

In order to solve this integral, the partial integral formula (11) will be used:

$$\int dx \ln(x^2 + w^2) = x\ln(x^2 + w^2) - \int dx(x\frac{2x}{x^2 + w^2}) \qquad (287)$$

The last term of Eq. (265) may be simplified according to the following steps:

$$\int dx(x\frac{2x}{x^2 + w^2}) = -2\int dx\frac{x^2 + w^2 - w^2}{x^2 + w^2} = -2\int dx + 2w^2\int dx\frac{1}{x^2 + w^2} \qquad (288)$$

which after integration thereby using Eq. (167) [14] gives the result

$$\int dx(x\frac{2x}{x^2 + w^2}) = -2x + 2w\arctan\frac{x}{w} \qquad (289)$$

Hence, $\vec{F}_4 = -½J^2(w^2)\int_{z_1=0}^{t} dz_1((x\ln(x^2+w^2) - 2x + 2w\arctan\frac{x}{w})\Big|_{x=-z_1}^{t-z_1})$

(290)

Inserting the values of x into the primitive function gives

$$\vec{F}_4 = -½J^2(w^2)\int_{z_1=0}^{t} dz_1((t - z_1)\ln((t - z_1)^2 + w^2) - 2(t - z_1) + 2w\arctan\frac{t - z_1}{w} - (-z_1)\ln((-z_1)^2 + w^2) +$$

$$-½J^2(w^2)\int_{z_1=0}^{t} dz_1(\ 2(-z_1) - 2w\arctan\frac{(-z_1)}{w}) \qquad (291)$$

simpler written:

$$\vec{F}_4 = -½J^2(w^2)\int_{z_1=0}^{t} dz_1((t - z_1)\ln((t - z_1)^2 + w^2) - 2t + 2w\arctan\frac{t - z_1}{w} + z_1\ln({z_1}^2 + w^2) + 2w\arctan\frac{z_1}{w})$$

(292)

This integral will now be solved for one term after another.
Before doing so, it would be practical to first simplify the integrand and put its respective terms on one row each, as will be done in the following treatment.

$$\vec{F}_{41a} = -½J^2(w^2)\int_{z_1=0}^{t} dz_1((t - z_1)\ln((t - z_1)^2 + w^2) \qquad (293)$$

$$\bar{F}_{41b} = -½J^2(w^2)\int_{z_1=0}^{t} dz_1(z_1 \ln(z_1{}^2 + w^2)) \quad (294)$$

$$\bar{F}_{42a} = -½J^2(w^2)\int_{z_1=0}^{t} dz_1(2w\arctan\frac{t-z_1}{w}) \quad (295)$$

$$\bar{F}_{42b} = -½J^2(w^2)\int_{z_1=0}^{t} dz_1(2w\arctan\frac{z_1}{w}) \quad (296)$$

$$\bar{F}_{43} = -½J^2(w^2)\int_{z_1=0}^{t} dz_1(-2t) \quad (297)$$

In order to solve Eq. (293) the following variable substitution will be done: $t - z_1 = x$. This makes the boarders change: $z_1 = 0$ is transformed to $x = t$ and $z_1 = t$ to $x = 0$ .Further one can rewrite $dz_1 = -dx$. Hence, one may now rewrite Eq. (293)

$$\bar{F}_{41a} = -½J^2(w^2)\int_{zx=t}^{0} (-dx)x\ln(x^2 + w^2) \quad (298)$$

Using now the partial integration formula (11), the last integral of Eq. (298) may be rewritten:

$$\int dx(x\ln(x^2 + w^2)) = \frac{x^2}{2}\ln(x^2 + w^2) - \int dx(\frac{x^2}{2}\frac{2x}{x^2 + w^2}) \quad (299)$$

The last term of Eq. (299) may be simplified according to the following steps:

$$-\int dx(\frac{x^2}{2}\frac{2x}{x^2 + w^2}) = -\int dx\frac{x(x^2 + w^2 - w^2)}{x^2 + w^2} = -\int dx(x) + w^2\int dx\frac{x}{x^2 + w^2} \quad (300)$$

Performing the integrations gives

$$-\int dx(\frac{x^2}{2}\frac{2x}{x^2 + w^2}) = -\frac{x^2}{2} + \frac{w^2}{2}\ln(x^2 + w^2) \quad (301)$$

and accordingly

$$\bar{F}_{41a} = -½J^2(w^2)((x^2/2)\ln(x^2+w^2) - (1/2)x^2 + (w^2/2)\ln(x^2+w^2))_{x=t(}^{0}$$

(302)

Inserting the values of x into the primitive function gives

$$\bar{F}_{41a} = -½J^2(w^2)(\frac{t^2}{2}\ln(t^2 + w^2) - \frac{t^2}{2} + \frac{w^2}{2}\ln(t^2 + w^2) - \frac{w^2}{2}\ln w^2) \quad (303)$$

Simpler written $d\ \bar{F}_{41a} = -½J^2(\frac{w^2t^2}{2}\ln(t^2 + w^2) + \frac{w^4}{2}\ln(t^2 + w^2) - \frac{w^4}{2}\ln w^2 - \frac{w^2t^2}{4})$

(304)

Eq. (294) reminds of Eq. (293). It may be recalled that integrating a function of z_1 from $0 \to t$ is equal to integrating the same function, but with argument $t - z_1$ from $0 \to t$.

Hence, $\bar{F}_{41b} = \bar{F}_{41a} = -½J^2(\frac{w^2t^2}{2}\ln(t^2 + w^2) + \frac{w^4}{2}\ln(t^2 + w^2) - \frac{w^4}{2}\ln w^2 - \frac{w^2t^2}{4})$ (305)

In order to solve Eq. (305) the following variable substitution will be done: $\frac{t - z_1}{w} = x$. This makes the boarders change: $z_1 = 0$ is transformed to $x = \frac{t}{w}$ and $z_1 = t$ to $x = 0$ .Further one can rewrite $dz_1 = -wdx$. Hence, one may now rewrite Eq. (305):

$$\vec{F}_{42a} = -\tfrac{1}{2}J^2(2w^4)\int_{x=t/w}^{0}(-wdx)(\arctan x) = -\tfrac{1}{2}J^2(2w^4)\int_{x=0}^{t/w}dx\arctan x \qquad (306)$$

Using now the integral formula (177) [15] gives

$$\vec{F}_{42a} = -\tfrac{1}{2}J^2(2w^4)((x\arctan x-(1/2)\ln(1+x^2))_{x=t/w}^{0} \qquad (307)$$

Inserting the values of x into the primitive function gives

$$\vec{F}_{42a} = -\tfrac{1}{2}J^2(2w^4)(\frac{t}{w}\arctan\frac{t}{w}-\frac{1}{2}\ln(1+(\frac{t}{w})^2) \qquad (308)$$

simpler written: $\vec{F}_{42a} = -\tfrac{1}{2}J^2(2w^3t\arctan\frac{t}{w}-w^4\ln(w^2+t^2)+w^4\ln w^2)$

(309)

Eq. (296) reminds of Eq. (295) It may be recalled that integrating a function of z_1 from $0\rightarrow t$ is equal to integrating the same function, but with argument $t-z_1$ from $0\rightarrow t$.

Hence, $\vec{F}_{42b} = \vec{F}_{42a} = -\tfrac{1}{2}J^2(2w^3t\arctan\frac{t}{w}-w^4\ln(w^2+t^2)+w^4\ln w^2)$

(310)

Eq. (297) is solved very easily, since the integrand is only a constant. Hence,

$\vec{F}_{43} = -\tfrac{1}{2}J^2(-2w^2t^2)$ (311)

In order to simplify the reader's overview, all the results of integrating the integral equations (293) to (297) will here be gathered

$$\vec{F}_4 = -\tfrac{1}{2}J^2(4w^3t\arctan\frac{t}{w}-w^2t^2+(w^2t^2+w^4-2w^4)\ln(t^2+w^2)+(2w^4-w^4)\ln w^2-2w^2t^2)$$

(312)

or even simpler: $\vec{F}_4 = -\tfrac{1}{2}J^2(4w^3t\arctan\frac{t}{w}-3w^2t^2+(w^2t^2-w^4)\ln(t^2+w^2)+w^4\ln w^2)$

(313)

Solving $\vec{F}_5$ (191), the final result given in expression(314)

This integral is very easily solved, since the integrand is only a constant.

Hence, $\vec{F}_5 = -\tfrac{1}{2}J^2(-3w^2t^2)$ (314)

Writing the total result $\vec{F} = \vec{F}_1+\vec{F}_2+\vec{F}_3+\vec{F}_4+\vec{F}_5$ (315)

The task is now to gather all the results attained above into one comprehensive expression, in the shape of a table, with all terms belonging to a certain function on one row each. The results were attained in the expressions (227), (263), (284), (313) and (314) respectively.

First, however, the results are gathered plainly in the order they are attained:

$$\vec{F} = -\tfrac{1}{2}J^2((\frac{4w^3t}{3}+4w^3t)\arctan\frac{w}{t}+\frac{8w^2t^2}{3}+2w^2t^2+\frac{4w^4}{3}\ln(t^2+w^2)-\frac{4w^4}{3}\ln w^2)$$

$$-\tfrac{1}{2}J^2(\frac{4w^3t}{3}\arctan\frac{t}{w}-\frac{5w^2t^2}{6}+\frac{7t^4}{36}+(\frac{-w^4}{2}-\frac{t^4}{6})\ln(t^2+w^2)+\frac{w^4}{2}\ln w^2)$$

$$-\tfrac{1}{2}J^2(\frac{t^4}{6}\ln t^2-\frac{7t^4}{36})$$
$$-\tfrac{1}{2}J^2(4w^3t\arctan\frac{t}{w}-3w^2t^2+(w^2t^2-w^4)\ln(t^2+w^2)+w^4\ln w^2)$$
$$-\tfrac{1}{2}J^2(-3w^2t^2))$$
$$-\tfrac{1}{2}J^2(-2w^3t\pi)) \qquad (316)$$

The table will be as follows, without any simplifications.:

$$\bar{F}=-\tfrac{1}{2}J^2\ ((\frac{4wt^3}{3}+4w^3t)\arctan\frac{w}{t}+(\frac{4w^3t}{3}+4w^3t)\arctan\frac{t}{w}$$
$$+\frac{8w^2t^2}{3}-\frac{5w^2t^2}{6}-3w^2t^2-3w^2t^2$$
$$-2w^3t\pi$$
$$+(\frac{4w^4}{3}-\frac{w^4}{2}-\frac{t^4}{6}+w^2t^2-w^4)\ln(t^2+w^2)$$
$$+\frac{7t^4}{36}-\frac{3t^4}{36}-\frac{4t^4}{36}$$
$$+(\frac{-4w^4}{3}+\frac{w^4}{2}+w^4)\ln w^2$$
$$+\frac{t^4}{6}\ln t^2) \qquad (317)$$

Further steps will now be undertaken in order to simplify the expression above,
At first, of course one can easily simplify by adding similar terms. This leads to:

$$\bar{F}=-\tfrac{1}{2}J^2\ ((\frac{4wt^3}{3}+4w^3t)\arctan\frac{w}{t}+(\frac{4w^3t}{3}+4w^3t)\arctan\frac{t}{w}-$$
$$-\frac{25w^2t^2}{6}-2w^3t\pi+(-\frac{w^4}{6}-\frac{t^4}{6}+w^2t^2)\ln(t^2+w^2)+\frac{w}{6}\ln w^2+\frac{t^4}{6}\ln t^2)$$
(318)

At first one may now remark the symmetry between two of the $\arctan$ terms:

$$4w^3t(\arctan\frac{w}{t}+\arctan\frac{t}{w}) \qquad (319)$$

Since $\arctan\frac{1}{z}=arc\cot z$ (320) [16]

and $\arctan z+arc\cot z=\pm\frac{1}{2}\pi$ (321) [17]

Where the positive sign constitutes the primary solution, which otherwise are attained if using the complex definitions of the arctan and arccot functions:

$\arctan z=\frac{i}{2}\ln\frac{i+z}{i-z}$ (322) [18] and $arc\cot z=\frac{i}{2}\ln\frac{z-i}{z+i}$ (323) [19]

Hence, expression (295) develops to $4w^3t\frac{\pi}{2}=2\pi w^3t$ (324)

$$\bar{F}=-\tfrac{1}{2}J^2\ (\ \frac{4wt^3}{3}\arctan\frac{w}{t}+\frac{4w^3t}{3}\arctan\frac{t}{w}+2\pi w^3t-2w^3t\pi-$$

$$-\frac{25w^2t^2}{6}+(-\frac{w^4}{6}-\frac{t^4}{6}+w^2t^2)\ln(t^2+w^2)+\frac{w^4}{6}\ln w^2+\frac{t^4}{6}\ln t^2) \qquad (325)$$

$$(-2w^3t\pi)$$

Since the current $I = Jwt$ (303) and further a suitable approximation is that the width w of the wire may be regarded as approximately the same as the thickness t, i.e. $w \cong t$ (326) Which altogether leads to

$$\bar{F} = -½I^2 (\frac{4}{3}\frac{\pi}{4}+\frac{4}{3}\frac{\pi}{4}-\frac{25}{6}+\frac{2}{3}\ln(2w^2)+\frac{1}{6}\ln w^2+\frac{1}{6}\ln w^2) \qquad (327)$$

This expression may be further simplified, and one finally attains:

$$\bar{F} = -½I^2 (\frac{2\pi}{3}-\frac{25}{6}+\frac{2}{3}\ln 2+\ln w^2) \qquad (328)$$

However, since the result is achieved for only one branch of the bridge, for the branch 10-7. Since branch 2-5 shows the same properties with respect to current and dimensions of the bridge, the result according to expression (305) can simply be doubled in order to attain the results from both branches. Hence, $F_s^{"} = 2d^2\bar{F} = I^2 (\frac{25}{6}-\frac{2\pi}{3}-\frac{2}{3}\ln 2-2\ln w)$ (329)

The sum of Eq. (329) and (128) gives the total force between the two halves of Ampère's Bridge according to the interpretation of Wesley. He writes:

$$\frac{F}{2I^2}=\frac{13}{12}-\frac{\pi}{3}-(\frac{2}{3})\ln 2+\sqrt{1+\frac{L^2}{M^2}}-\ln(1+\sqrt{1+\frac{L^2}{M^2}})+\ln(L/w) \qquad (330) \qquad [3]$$

Wesley also prefers to use the circular cross section d of the wire that Ampére's Bridge consists of instead of the Cartesian variable w due to a quadratic cross section. That makes the following transformation formula necessary:

$$w=\frac{\sqrt{\pi}d}{2} \qquad (331)$$

4. Conclusions concerning the derivation by Wesley

It has been convincingly shown that Dr. Wesley has correctly attained his formula, derived from Ampère's Law, intended at predicting the force between two parts of Ampère's Bridge. Measurements on Ampere's Bridge performed by Pappas and Moyssides [24] has been presented in his papers [3] and [20].and he has succeeded in applying his formula upon that set, thus achieving some resemblance with the measurement results by Pappas and Moyssides [24]. However, Jonson has pointed to deficits in that result [6] and proposes another model, making use of only Coulomb's law.

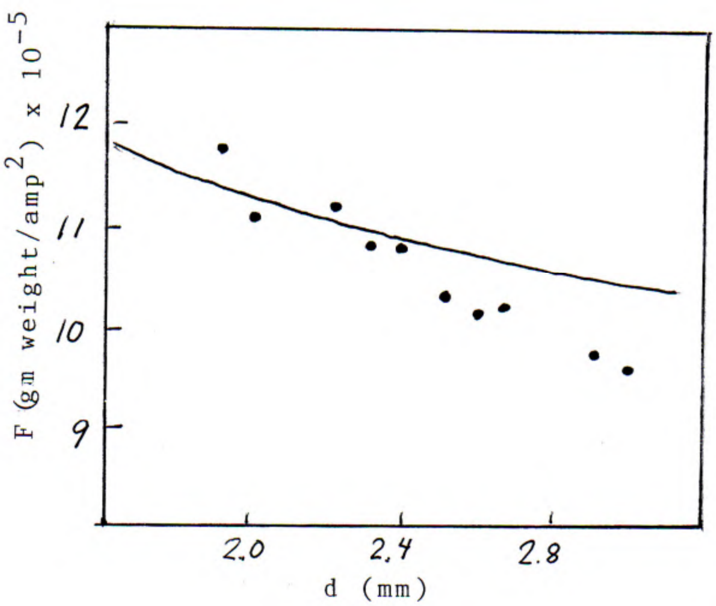

Figure 2. Showing the force between the two halves of Ampère's Bridge as a function of the circular cross section d [5]. (Figure by Wesley redrawn by Jonson)

5. Jonson's method use Coulombs law in order to predict the force within Ampère's Bridge.

It has usually been assumed that Coulomb's law, which is normally being used in order to explore electrostatic forces, is unable to account for electromagnetic forces, i.e. especially forces between electric currents.

Jonson has made an effort to do that [6], thereby taking into account the effects of the time delay that inevitably occurs with respect to all action-at-a distance. (He claims support in favour of the opinion that believes that no action takes place instantly, that is requires a transport time in order to have effect).

5.1. Definition of variables that appear in the expressions for the force upon Ampère's Bridge.

Please see chapter 3.1. That is, the same variables are defined by Jonson as by Wesley.

5.2. The expression for the force between two currents according to Jonson.

The same procedure with respect the choice of integral type is used as in the analysis by Wesley.

For current elements far away from each others line integrals will be used and

$$d^2\vec{F} = \frac{\mu_0}{4\pi}\frac{I^2\vec{R}(d\vec{s}_1 \bullet \vec{R})(d\vec{s}_2 \bullet \vec{R})}{R^5} \qquad (332)$$

and for parts of the bridge in close contact with each others volume integrals have to be used:

$$d^6\vec{F} = \frac{\mu_0}{4\pi}\frac{(\vec{J}_1 \bullet \vec{R})(\vec{J}_2 \bullet \vec{R})}{R^5} \qquad (333)$$

Both formulas can straightforwardly be derived from the result in an earlier paper on the subject [8].

The formula is there written according to:

$$\frac{d^2\vec{F}}{dx_1dx_2} = \frac{(-\rho_1\vec{v}_1 \bullet \vec{R})}{Rc}\frac{(-\rho_2\vec{v}_2 \bullet \vec{R})}{Rc}\frac{\vec{u}_R}{4\pi\varepsilon_0 R^2} \quad (334)$$

where $I = \rho v$ (335) is being used.

The well-known relation $c^2 = \dfrac{1}{\mu_0\varepsilon_0}$ (336)

makes the identification with Eq. (332) and (333) easily conceivable.

In order to attain formal expressions for the force due to Coulomb's Law, expressed in this application, a comparison between Wesley's expressions for Ampère's Law and Jonson's for Coulomb's law must be done. They show one important feature: Both contain terms proportional to $\dfrac{(d\vec{s}_1 \bullet \vec{R})(d\vec{s}_2 \bullet \vec{R})\vec{R}}{R^5}$ (337)

The expressions by Wesley were earlier defined in Eq. (1) and (2). And are repeated here for the reader's convenience:

$$\frac{d^6\vec{F}}{d^3r_2d^3r_1} = \vec{r}(-2\frac{\vec{J}_2 \bullet \vec{J}_1}{r^3} + 3\frac{(\vec{J}_2 \bullet \vec{r})(\vec{J}_1 \bullet \vec{r})}{r^5}) \quad (1)$$

$$d^2\vec{F} = I_2 I_1 \vec{r}(-2\frac{(d\vec{s}_2 \bullet d\vec{s}_{1)}}{r^3} + 3\frac{(d\vec{s}_2 \bullet \vec{r})(d\vec{s}_1 \bullet \vec{r})}{r^5}) \quad (2)$$

Seemingly, Jonson uses only the second type of term and with a constant three times smaller than Wesley. Hence, in order to attain the Jonson expression, firstly the first type of term has to be removed from all the derivations by Wesley, secondly, the remaining expression must accordingly be divided by three.

Also ***a third thing*** must be done, which has thus far been unrecognized by everybody: to take into account the impact of the current source (battery) of the circuit.

This was originally done in the 1997 paper of this author [7]. here, only the results will be used.

A difference between the expressions by Wesley and Jonson is also the different way they define the coupling constant before the spatial equations: The experiment reports were reported [1] with the unit abampere instead of ampere, which causes a need to divide the whole force by 100 (10 for each current) and they used gramweight for the force, which in turn makes a division by $980mm/s^2$ needed. However, Jonson follows the scaling procedure by Wesley, when dealing with the Coulomb model of his. But when the magnetic force law is treated, the commonplace $\dfrac{\mu_0}{4\pi} = 10^{-7}$ is being used as a coupling constant.

(338)

5.2.1. The removal of the 'first type' terms from the expressions by Wesley.

The following terms are to be removed: Eq. (34), Eq. (69) twice, since section 3.1.3 describes an equal case to section. 3.1.2. Finally, Eq. (138) must be removed.from Eq. (158).

If all this is correctly being done, all terms due to the 'first' Ampere law term disappears. Wesley's approach to separate the force into two main terms, one due to portions of the bridge not in immediate contact with each others, F', and portions in immediate contact, F'', is used also with respect to Coulomb's law interpreted by Jonson, but then with an added letter J to the variable: F_J' and F_J'' instead.

The result of these actions is that a new expression for the force term due to parts of the bridge in close contact with each other (i.e. $10 \to 7$ and $2 \to 5$), corresponding to Eq. (138), according to: (Twice the result from Eq. (138) is needed, since $10 \to 7$ and $2 \to 5$ both contribute with equal results due to their symmetric position (also commented below Eq. (158)).

$$F''_J = 2(I^2(-1-3\ln\frac{M}{M-N}+3\ln 2N - 3J^2\int_{z_2=0}^{t} dz_2 \int_{x_2=0}^{w} dx_2 \int_{z_1=0}^{t} dz_1 \int_{x_1=0}^{w} dx_1 \ln((x_2-x_1)^2+(z_2-z_1)^2)))$$

(339)

In order to attain F_J', Eq. (34) and twice Eq. (69) is removed from Wesley's expression (Eq. (128), the terms thus being

$$I^2(4\sqrt{1+(\frac{L}{M})^2}-4+4\ln(\frac{M-N+\sqrt{(M-N)^2+L^2}}{L}\frac{N+\sqrt{N^2+L^2}}{M+\sqrt{M^2+L^2}})) \qquad (340)$$

which gives:

$$F_J' = 2I^2(-\sqrt{1+(\frac{L}{M})^2}+2-\ln\frac{M-N}{M}-\ln\frac{N}{L}-\ln(1+\sqrt{1+(\frac{L}{M})^2})-2\ln\frac{M-N+\sqrt{(M-N)^2+L^2}}{L}-$$
$$-2\ln\frac{N+\sqrt{N^2+L^2}}{M+\sqrt{M^2+L^2}}) \qquad (341)$$

Adding now Eq. (340) to Eq. (341), thereafter dividing by a factor 3, gives the parts of Ampère's law that according to Jonson corresponds to Coulomb's law:

$$F_J = \frac{1}{3}2I^2(-\sqrt{1+(\frac{L}{M})^2}+2-\ln\frac{M-N}{M}-\ln\frac{N}{L}-\ln(1+\sqrt{1+(\frac{L}{M})^2})-2\ln\frac{M-N+\sqrt{(M-N)^2+L^2}}{L}-$$
$$-2\ln\frac{N+\sqrt{N^2+L^2}}{M+\sqrt{M^2+L^2}})+$$
$$+2(I^2(-\frac{1}{3}-\ln\frac{M}{M-N}+\ln 2N)-2J^2\int_{z_2=0}^{t} dz_2 \int_{x_2=0}^{w} dx_2 \int_{z_1=0}^{t} dz_1 \int_{x_1=0}^{w} dx_1 \ln((x_2-x_1)^2+(z_2-z_1)^2)))$$

(342)

The last term, however, is equal to Eq. (329), whose result may simply be added.(repeated here for convenience)

$$I^2(\frac{25}{6}-\frac{2\pi}{3}-\frac{2}{3}\ln 2-2\ln w) \qquad (329).$$

Using also $w=\frac{\sqrt{\pi}d}{2}$ (331)

Makes it easy to evaluate the formula, favourably for the endpoint values of the diagram, i.e. $1.6mm$ and $3.2mm$ respectively, which gives – after taking also into account Eq. (349), the values $F_J = 10.7(gmweight/amp^2)\times 10^{-5}$ and $F_J = 9.3(gmweight/amp^2)\times 10^{-5}$.

Apparently, this formula too (i.e. Coulomb's law in Jonson's interpretation) succeeds in predicting a decreasing force with respect to the circular cross-section, the values being situated not-so-far from the measured ones.

5.3. The impact of the current source in contributing to the force.

Jonson also discovered that the very current (voltage) source also plays a role in contributing to the total force between two currents [7]

The correction to be made is due to the effects of the current source, since all the work that is being done on the electrons going through the circuit by the electric field must be ***exactly***

balanced by the work being done by the same electric field on the electrons which remain on the poles. As the current flows, namely, the electric field inevitably weakens, and hence, the electrons that still are situated on the poles feel free to move slightly. How this is best mathematically treated is described in the 1997 paper by this author [7].
The current to be defined to the poles is the equivalent of the length of the circuit times the current through the circuit times a dirac function due to the 'point' nature of the 'slight' movement of the pole electrons, while still being at the poles. For simplicity, the battery is assumed to be situated at half the length of branch 1. Hence, the 'pole current'

$$I_{POLE} = 2(L+M)I\delta(x_1 - \frac{L}{2}) \qquad (343)$$

Three contributions to the total force will appear: to branch 5,6 and 7 espectively.

5.3.1 Correction term battery to branch 6.

The following expression for the force may be defined in this case, thereby using Eq. (332):

$$I^2 \int_{x_2=0}^{L} dx_2 \int_{x_1=0}^{L} dx_1 \frac{(-I(2(L+M)\delta(x_1-\frac{L}{2})(-Idx_2))M^2)}{((x_2-x_1)^2+M^2)^{5/2}} \qquad (344)$$

Performing the integrals gives :

$$\frac{2}{3} I^2 M(L+M)(\frac{-L}{((\frac{L}{2})^2+M^2)^{\frac{3}{2}}} + \frac{1}{M^2}\frac{2L}{\sqrt{L^2+M^2}}) \qquad (345)$$

5.3.2 Correction term battery to branch 7.

Again applying Eq. (332) now gives the following expression for the force:

$$I^2 \int_{x_2=0}^{L} dx_2 \int_{y_2=N}^{M} dx_1 \frac{(-x_1 y_2{}^2)(2(L+M))\delta(x_1-\frac{L}{2})dx_1 dy_2)}{(x_1{}^2+y^2)^{5/2}} \qquad (346)$$

Performing the integrals gives:

$$\frac{2}{3} I^2 L(L+M)(\frac{M}{((\frac{L}{2})^2+M^2)^{\frac{3}{2}}} - \frac{N}{((\frac{L}{2})^2+N^2)^{3/2}} + \frac{2}{L^2}\frac{N}{\sqrt{(\frac{L}{2})^2+N^2}} - \frac{2}{L}\frac{M}{\sqrt{(\frac{L}{2})^2+M^2}})$$

(347)

5.3.3 Correction term battery to branch 5.

Due to the symmetry of the problem, the same result will appear in this case.

5.3.4 The sum of all correction terms

Adding the results above, Eq. (345), double Eq. (346) makes:

$$I^2 \frac{L+M}{6}(\frac{-LM}{((\frac{L}{2})^2+M^2)3/2} + (\frac{2L}{M}-\frac{2M}{L})\frac{1}{\sqrt{(\frac{L}{2})^2+M^2}} - \frac{NL}{((\frac{L}{2})^2+N^2)^{3/2}} + \frac{2N}{L}\frac{1}{\sqrt{((\frac{L}{2})^2+M^2}})$$

(348)

Evaluation gives $F_{corr} \cong -0.51 I^2$ (349)

Using Eq. (342) and taking into account Eq. (349) gives a result that fairly well fits with measured values. 10.7×10^{-5} at the left side of the diagram, 9.3×10^{-5} at the right side, thereby using Wesley's scaling.

6. Comparison with the Magnetic Force Law (Lorentz Force)
Jonson has also performed a calculation of the force based upon the traditional 'Magnetic Force Law', the so-called Lorentz force [10].
In the DC cases the traditional expressions for the so-called magnetic field and the magnetic 'Lorentz' force will be used. The undergraduate course book this author has used during his MSc studies [22] expressed the laws as follows (indices adjusted to fit with this paper):

$$\vec{F}_m = \int I_2 d\vec{s}_2 \times \vec{B} \qquad (350)$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I_1 d\vec{s}_1 \times \vec{r}}{r^3} \qquad (351)$$

Now the contribution to the magnetic field at each branch of the upper bridge will be derived, provided it gives rise to a force component along the y axis. It can easily be stated which parts of the bridge will not give rise to y components, namely those aligned with the y axis, i.e. branch 5 and 7. Hence, the task before us is to derive expressions for the magnetic field at branch 6 only. All the three branches of the lower part of the bridge will contribute.

6.1. Magnetic field and magnetic force from branch 1 to 6.
Realizing first that $d\vec{s}_1 = (dx_1, 0, 0)$ (352) , $\vec{r} = (x_2 - x_1, M, 0)$ (353)
Using thereafter Eq. (324) gives:

$$\vec{B}_{1\to 6} = \frac{\mu_0}{4\pi}\int_{x_1=0}^{L} dx_1 \frac{MI_1\vec{u}_z dx_1}{((x_2 - x_1)^2 + M^2)^{3/2}} \qquad (354)$$

Using the variable substitution $M \tan\alpha = x_2 - x_1$ (355) gives $dx_1 = -\frac{Md\alpha}{\cos^2\alpha}$ (356)

This leads to $\vec{B}_{1\to 4} = \frac{\mu_0}{4\pi}\int_{x=x_2}^{x_2-L} d\alpha \frac{-M^2 I_1\vec{u}_z}{\cos^2\alpha}\frac{1}{(M^2\tan^2\alpha + M^2)^{3/2}}$ (357)

which reduces to

$$\vec{B}_{1\to 6} = \frac{\mu_0}{4\pi}\frac{-I_1\vec{u}_z}{M}\int_{\alpha=\arctan(x_2/M)}^{\arctan((x_2-L)/M)} d\alpha \;\cos\alpha \qquad (358)$$

and, finally, after performing the integral,

$$\vec{B}_{1\to 6} = \frac{\mu_0}{4\pi}\frac{-I_1\vec{u}_z}{M}\left(\frac{x_2 - L}{\sqrt{(x_2 - L)^2 + M^2}} - \frac{x_2}{\sqrt{(x_2)^2 + M^2}}\right) \qquad (359)$$

Thereafter Eq. (350) has to be used in order to attain the magnetic force $\vec{F}_m$
Realizing first that $ds_2 = (-dx_2, 0, 0)$ (360)
gives

$$\vec{F}_{m,1\to 6} = \frac{\mu_0}{4\pi}\frac{I^2}{M}\vec{u}_y \int_{x_2=0}^{L} dx_2\left(\frac{x_2}{\sqrt{(x_2)^2 + M^2}} - \frac{x_2 - L}{\sqrt{(x_2 - L)^2 + M^2}}\right) \qquad (361)$$

Solving the integral gives accordingly:

$$\vec{F}_{m,1\to 6} = \frac{\mu_0}{4\pi}\frac{I^2}{M}\vec{u}_y\left(\left(\sqrt{(x_2)^2 + M^2} - \sqrt{(x_2 - L)^2 + M^2}\right)_{x_2=0}^{L}\right)$$

(362)
Simpler, one may write

$$\vec{F}_{m,1\to 6} = \frac{\mu_0}{4\pi}\frac{2I^2}{M}\vec{u}_y\left(\sqrt{1 + \left(\frac{L}{M}\right)^2} - 1\right) \qquad (363)$$

A detailed derivation will follow here later, Short-to-speak, however, that force law fails completely to predict the 'parallel force', first observed by Ampere.

6.2. Magnetic field and magnetic force from branch 2 to 6.

Realizing first that $d\vec{s}_1 = (0, dy_1, 0)$ (364) , $\vec{r} = (x_2 - L, M - y_1, 0)$ (365)
Using thereafter Eq. (351) gives:

$$\vec{B}_{2\to 6} = \frac{\mu_0}{4\pi} \int_{y_1=0}^{N} dx_1 \frac{-(x_2 - L) I_1 \vec{u}_z dy_1}{((x_2 - L)^2 + (M - y_1)^2)^{3/2}} \quad (366)$$

Using thereafter the variable substitution $x = M - y_1$ (367) gives $dx = -dy_1$ (368)
This leads to:

$$\vec{B}_{2\to 6} = \frac{\mu_0}{4\pi} \int_{x=M}^{M-N} dx \frac{(x_2 - L) I_1 \vec{u}_z}{((x_2 - L)^2 + x^2)^{3/2}} \quad (369)$$

Using now the variable substitution $(x_2 - L)\tan\alpha = x$ (370)

gives $dx = \dfrac{(x_2 - L) d\alpha}{\cos^2 \alpha}$ (371)

Using all this makes it possible to rewrite Eq. (369):

$$\vec{B}_{2\to 6} = \frac{\mu_0}{4\pi} \int_{x=M}^{M-N} d\alpha \frac{(x_2 - L)^2 I_1 \vec{u}_z}{\cos^2 \alpha} \frac{1}{((x_2 - L)^2 + (x_2 - L)^2 \tan^2 \alpha)^{3/2}} \quad (372)$$

which reduces to

$$\vec{B}_{2\to 6} = \frac{\mu_0}{4\pi} \frac{I_1 \vec{u}_z}{x_2 - L} \int_{\alpha=\arctan M/(x_2 - L)}^{\arctan(M-N)/(x_2 - L)} d\alpha \quad \cos\alpha \quad (373)$$

and, finally, after performing the integral,

$$\vec{B}_{2\to 6} = \frac{\mu_0}{4\pi} \frac{I_1 \vec{u}_z}{x_2 - L} \left(\frac{M - N}{\sqrt{(M - N)^2 + (x_2 - L)^2}} - \frac{M}{\sqrt{M^2 + (x_2 - L)^2}} \right) \quad (374)$$

Realizing first that $d\vec{s}_2 = (-dx_2, 0, 0)$ (360)
gives

$$\vec{F}_{m,2\to 6} = \frac{\mu_0}{4\pi} (I_1 \vec{u}_y) \int_{x_2=0}^{L} dx_2 \frac{1}{x_2 - L} \left(\frac{M - N}{\sqrt{(M - N)^2 + (x_2 - L)^2}} - \frac{M}{\sqrt{M^2 + (x_2 - L)^2}} \right)$$
(375)

Using now the variable substitution $x_2 - L = \dfrac{1}{x}$ (376) giving thus $dx_2 = -\dfrac{dx}{x^2}$ (377)

makes it possible to rewrite the integral:

$$\vec{F}_{m,2\to 6} = \frac{\mu_0}{4\pi} (I_1 \vec{u}_y) \int_{x=-1/L}^{\infty} \frac{-dx}{x^2} x \left(\frac{M - N}{\sqrt{(M - N)^2 + (\frac{1}{x})^2}} - \frac{M}{\sqrt{M^2 + (\frac{1}{x})^2}} \right) \quad (378)$$

Which may be simplified to:

$$\vec{F}_{m,2\to 6} = \frac{\mu_0}{4\pi}(-I_1\vec{u}_y)\int_{x=-1/L}^{\infty} dx\left(\frac{1}{\sqrt{x^2+(\frac{1}{M-N})^2}} - \frac{1}{\sqrt{x^2+(\frac{1}{M})^2}}\right) \qquad (379)$$

Solving the integral gives:

$$\vec{F}_{m,2\to 6} = \frac{\mu_0}{4\pi}(-I_1\vec{u}_y)(\ln(x+\sqrt{x^2+(1/(M-N)^2)})\Big|_{x=-1/L}^{\infty} \qquad (380)$$

Finally, one attains

$$\vec{F}_{m,2\to 6} = \frac{\mu_0}{4\pi}(-2I^2\vec{u}_y)\ln\frac{-\frac{1}{L}+\sqrt{(\frac{1}{L})^2+(\frac{1}{M})^2}}{-\frac{1}{L}+\sqrt{(\frac{1}{L})^2+(\frac{1}{M-N})^2}} \qquad (381)$$

6.3. Magnetic field and magnetic force from branch 10 to 6.

Due to the symmetry of the result is the same as that from branch 2 to 6.

6.4. The total magnetic force.

The total magnetic force is attained as the sum of the three contributions above.
Using the values given by Wesley [5], [24], L=0.48 m, M=1.20 m and N= 0.43 m gives a resulting magnetic force $\vec{F}_m \cong 0.29\times 10^{-5}\vec{u}_y$ (382)
if using Wesley's scaling.

Apparently, the force is attractive but constant, with no dependence of the thickness of the circuit. Hence, the so-called magnetic (Lorentz) force is completely unable to account for the behaviour of the force within Ampère's bridge.

7. Conclusions.

A has already been stated, the Jonson results basically confirm the slope of the measurement dependence of the cross section of the wire that Ampère's Bridge consists of, whereas Wesley partially fails in this respect. However, the Lorentz force is completely unable to predict any spatial dependence.
It is also necessary to be stated that the strength of the current one measures, presumes the Lorentz force participating in the torque that forces the arrow of a volt (ampere) meter to move.

$\vec{\tau} = \vec{r}\times\vec{F}_m$ (383)

Hence, any measurement thus far contains a 'circular proof moment .

8. Variable list.

This remains to be done. Mostly, the variables being used are identical with those defined by Wesley [1], [20]. ***A complete list will appear rather soon.***

9. References.

[1] J.P. Wesley: 'Progress in Space-Time Physics', Benjamin Wesley, 1987
[1b] ibid p. 170
[2] p. 174
[3] p. 175
[4] ibid p. 173, Eq. (3)
[5] ibid p 177
[6] Jonson, J. O., "The Magnetic Force between Two Currents Explained Using Only Coulomb's Law", Chinese Journal of Physics, VOL. 35, NO. 2, April 1997, p. 139-149,

http://psroc.phys.ntu.edu.tw/cjp/find_content.php?year=1997&vol=35&no=2
[7] ibid. p 144
[8] ibid p. 143, Eq. (24).
[9] ibid p. 146, Fig. 4
[10] ibid p. 147
[11] Jonson NPA 2009
[12]] A. M. Ampère, "Mémoire. Sur la théorie mathématique des phénomènes électrodynamiques uniquement déduite de l'expérience, dans lequel se trouvent réunis les Mémoires que M. Ampère a communiqués à l'Académie royale des Sciences, dans les séances des 4 et 26 décembre 1820, 10 juin 1822, 22 décembre 1823, 12 septembre et 21 novembre 1825", 'Mémoires de l'Académie Royale des Sciences de l'Institut de France Année 1823, Tome VI, Paris, chez Firmin Didot, Père et fils, libraires, Rue Jacob, No 24, pp 175-387. Copy belonging to the Stockholm University Library (http://www.sub.su.se)
[13] Abramowitz, M. and Stegun, A., 'Handbook of Mathematical Functions, Dover Publications, Inc., New York, 1972, p. 11, Eq. 3.3.3
[13b] ibid p. 13 Eq. 3.3.40
[14] ibid p. 12 Eq. 3.3.21
[15] ibid p. 82 Eq. 4.4.60
[16] ibid p. 79 Eq. 4.4.8
[17] ibid. p. 79 Eq. 4.4.5
[18] Thomas Josefsson, 'Formel- och tabellsamling i matematik', Studentlitteratur, 1976, Lund, Sweden, ISBN 91-44-09014-5, p. 9 Eq. (38)
[19] ibid. Eq. (39)
[20] J. P. Wesley, Spec. Sci. Tech. **10,** 47 (1987)
[21] 'The Magnetic Force between Two Currents Further Analysed Using Coulomb's Law and Special Relativity Theory', Jan Olof Jonson, NPA conference paper 2009, Figure 5. yet to be published in the coming Proceedings volume.
[22] Erik Hallén, 'Elektricitetslära' (Eng. 'Electricity'), Almqvist & Wiksells Boktryckeri Aktiebolag, Uppsala 1968, Sweden, p. 111, Eq, (11,21) and (11,22).
[23] Christine Blondel, 'A.-M. Ampère et la création de l'électrodynamique (1820-1827)', Mémoires de la section des sciences 10, Paris Bibliothèque nationale, 1982, p 132
[24] Pappas, Moyysides 'Rigorous quantitative test of Biot-savart-Lorentz forces', J. Appl. Phys. **59**,1 January 1986 (19)