Cochetkov's Speeding Bola: Yet Another Entanglement for Special Relativity

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For a "bola" with identical masses rotating freely in space, a Lorentz-boosted observer sees a time-varying momentum and kinetic energy, in violation of special-relativistic momentum and energy conservation laws. In the moving frame, the bola string is curved, not straight, and thrums with a period that is half the Lorentz-time-dilated period of the bola in the CM frame.

1. Introduction

Because of its frame-dependence of simultaneity, special relativity is vulnerable when used for extended mechanical systems---not just rigid systems that are ruled inadmissible because of their presumed infinite signal speeds, but quantum systems in which such superluminal signal speeds in one frame (entanglement) are endorsed and yet lead to paradoxical backward-time travel in another frame. Now, V. N. Cochetkov [1, 2] poses a paradox based on an extended system comprising two identical masses m connected by a massless string that revolve about their center of mass (CM), executing in the CM frame a uniform circular motion of radius R and angular frequency ω. The object resembles a bola---a weapon that is thrown to entangle the legs of animals. The bola is a metaphor for entanglement, hence the title of this paper. My analysis below is simpler than Cochetkov's.

2. Derivation of Paradox

To show the bola paradox, I adopt dimensionless quantities $x \leftarrow x/R$, $y \leftarrow y/R$, $t \leftarrow ct/R$, $\omega \leftarrow R\,\omega/c$. The motion of particles 1 and 2 in the CM frame is

$$x_1 = \cos(\omega t),\ \ y_1 = \sin(\omega t),\ \ x_2 = -\cos(\omega t),\ \ y_2 = -\sin(\omega t). \quad (1)$$

Lorentz-transforming these quantities to incur boost v in the x direction:

$$x_1' = \gamma\,[\cos(\omega t_1) - v t_1]\,,\ \ y_1' = \sin(\omega t_1) \quad (2a)$$

$$x_2' = \gamma\,[-\cos(\omega t_2) - v t_2]\,,\ \ y_2' = -\sin(\omega t_2) \quad (2b)$$

where $\gamma = (1 - v^2)^{-0.5}$. The time variables t are subscripted because they depend on position in the moving frame but are evaluated at the same instant t′ in that frame. By inverse Lorentz transformation:

$$t_1 = \gamma\,(t' + v\,x_1'),\ \ t_2 = \gamma\,(t' + v\,x_2')\,. \quad (3)$$

The time variables t are subscripted because they depend on position in the moving frame when evaluated at the same instant t′ in that frame.

Now evaluate $u_{1x}' = dx_1'/dt'$ etc. using the chain rule on Eqs. 2 and 3:

$$u_{1x}' = \gamma\,[-\omega \sin(\omega t_1) - v]\,\gamma\,(1 + v\,u_{1x}') \quad (4a)$$

$$u_{1y}' = \omega \cos(\omega t_1)\,\gamma\,(1 + v\,u_{1x}') \quad (4b)$$

$$u_{2x}' = \gamma\,[\omega \sin(\omega t_2) - v]\,\gamma\,(1 + v\,u_{2x}') \quad (4c)$$

$$u_{2y}' = -\,\omega \cos(\omega t_2)\,\gamma\,(1 + v\,u_{2x}'). \quad (4d)$$

Solve Eqs. 4a and 4c for u_{1x}' and u_{2x}':

$$u_{1x}' = -\,[\omega \sin(\omega t_1) + v]/[1 + v\,\omega \sin(\omega t_1)] \quad (4a')$$

$$u_{2x}' = -\,[-\omega \sin(\omega t_2) + v]/[1 - v\,\omega \sin(\omega t_2)]. \quad (4c')$$

Momentum will be conserved in the moving frame only if the total momentum (divided by m) **p**' is independent of time t′, where

$$\mathbf{p}' = \mathbf{u_1}'/(1 - u_1'^2)^{0.5} + \mathbf{u_2}'/(1 - u_2'^2)^{0.5} \quad (5)$$

where $\mathbf{u}_i' = (u_{ix}', u_{iy}')$, $u_i' = (u_{ix}'^2 + u_{iy}'^2)^{0.5}$, and $i = 1, 2$.

To test momentum conservation, we substitute Eqs. (1)-(4) into Eq. (5) for several values of t′. First insert Eq. (3) into the x equations of Eq. (2) and simplify using the definition of γ:

$$\gamma(x_1' + vt') = \cos(\gamma\omega v x_1' + \gamma\omega t') \quad (6a)$$

$$\gamma(x_2' + vt') = -\cos(\gamma\omega v x_2' + \gamma\omega t'). \quad (6b)$$

Then, given numerical values for t', ω, and v (hence γ), use the following algorithm:

1. Numerically solve Eqs. (6) for x_1' and x_2'. Note $x_2' = -x_1'$ when t' = 0, but not otherwise. [Note: Each of Eqs. (6) has the form $x' = f(x')$, and in this case the iteration $x'_{new} = f(x'_{old})$, starting with $x'_{old} = 0$, converges satisfactorily.]
2. Substitute the now-known values x_1', x_2' in Eq. 3, thereby retrieving t_1, t_2.
3. Substitute t_1 and t_2 into Eqs. (4a′), (4b), (4c′), (4d) above, to retrieve u_{1x}', u_{1y}', u_{2x}', u_{2y}'.
4. Evaluate Eq. (5) (x and y momentum components), and see if they change in t'.

As an example, let v = 0.6 and w = 0.5. Then,

for t′ = 0, **p′** = (-2.1416, 0);

for t′ = 1, **p′** = (-2.0902, -0.1115); and

for t′ = 2, **p′** = (-1.9542, -0.1677).

Because **p′** depends on t′, momentum is not conserved in the moving frame, contrary to what is desired in special relativistic dynamics. For another example (v = 0.8, ω = 0.7, chosen to be more extreme because otherwise the curves look like sine waves) Fig. 1 shows a plot versus t′ of the momentum ($p_x' + 2v\gamma$, p_y') and kinetic energy

$$T = 1/(1 - u_1'^2)^{0.5} + 1/(1 - u_2'^2)^{0.5} - 2. \quad (7)$$

Here, p_x' is offset to compensate the CM momentum, both for graphical convenience and to show that the time-averaged total momentum is not the CM momentum $-2v\gamma$. From Fig. 1, we can see that all three quantities are periodic, non-sinusoidal, and non-constant in t′.

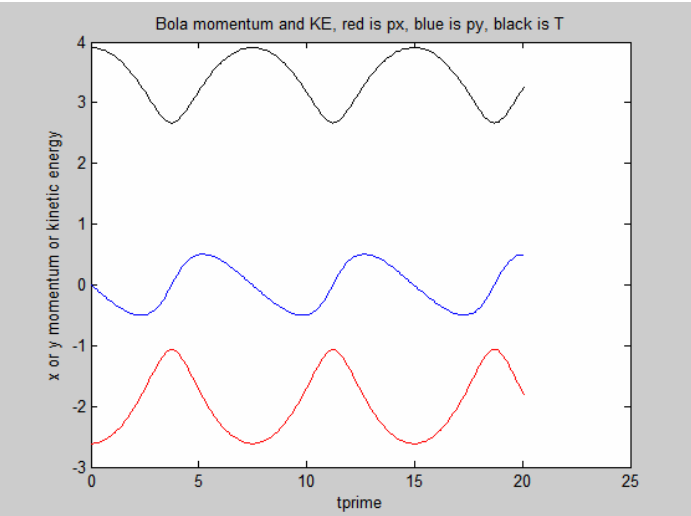

Figure 1. Mass-normalized bola T (top curve), p_y' (middle curve) and $p_x' + 2v\gamma$ (bottom curve) as functions of t′, for v = 0.8, ω = 0.7

The periodicity follows analytically: Consider the time translation $t' \rightarrow t' + \pi\gamma/\omega$ and hypothesized space translation $x' \rightarrow x' - v\pi\gamma/\omega$ and $y' \rightarrow y'$. This substitution produces the mapping $\omega\gamma(t' + v x') \rightarrow \omega\gamma(t' + v x') + \pi$, which reverses the sign of the right-hand sides of Eq. 6. Also, it produces the identity mapping $(x' + vt') \rightarrow (x' + vt')$ on the left-hand sides of Eq. 6. The reversed sign on the right-hand sides is compensated by the exchange symmetry between the two masses. (The other mass is in the non-sign-reversed position at time $t' + \pi\gamma/\omega$.) Therefore, with particle labels inserted, the dynamics dictated by Eq. 6 are invariant under the time translation $t' \rightarrow t' + \pi\gamma/\omega$.

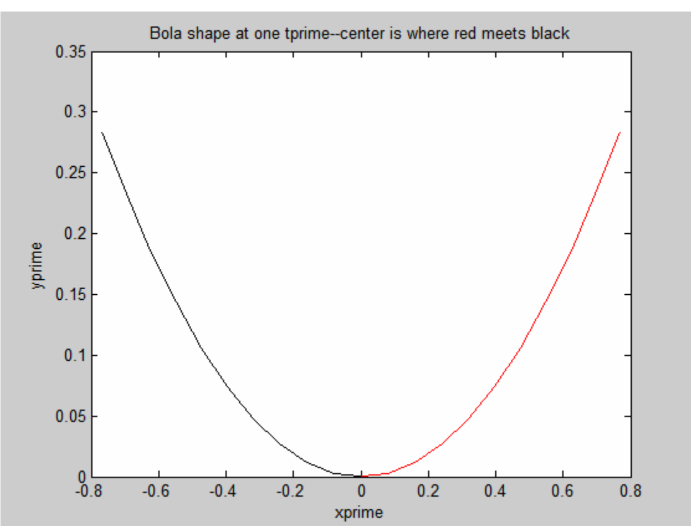

Figure 2. Bola shape at t′ = 0, for v = 0.6, ω = 0.5

It is also the case that the bola string, although straight in the CM frame, is curved in the moving frame (see Fig. 2).[1] In common with p′ and E′, the shape of the bola string thrums with the period of $\gamma\pi/\omega$, which is ½ the time-dilated period of the bola. (As above, the factor ½ comes from the identical masses m exchanging roles in the middle of a bola period.)

3. Conclusion

This paradox emerges from an unexplored area of special relativity: The time-dependence of total momentum in a spatially extended system after a Lorentz boost is applied that changes the definition of simultaneity. Other related paradoxes also exist: e.g. [3] posit a one-dimensional box with identical point masses synchronously and elastically hitting each other and opposite ends of the box. From a moving frame, special-relativistic kinematics says the center of mass CM of the box (exclusive of the point masses) moves uniformly, but momentum conservation says CM moves jerkily. The ends of the box do not even have to be connected to each other for this paradox to work. As with the bola, a kinematical Lorentz transformation leads to a failure of momentum conservation.

A simile from Tom Phipps seems particularly apt: "The special theory is a very special theory indeed. […] Like an over-bred race horse, it seems apt to break a leg in the first chuck-hole if let into an ordinary, real-world pasture."[4] Phipps's simile plays fortuitously with the leg-entangling bola of the present paper.

That said, not all of special relativity need be lost due to this paradox. Left with a choice of whether to keep momentum conservation or special-relativistic kinematics, the former seems more palatable. Lorentz transformations may apply to energy-momentum 4-vectors and such, but a new kinematic transformation will have to be found for spacetime itself.

The purpose of a paradox is to plant a question. If someone can resolve the question by a more "complete" analysis, let him do so. No easy path exists out of this paradox, but the burden of proof is on advocates of special relativity.

Acknowledgment

I thank Thomas E. Phipps, Jr. for many illuminating discussions of this work, and also I thank Victor N. Cochetkov for his collegial correspondence in the past three months.

References

[1] V. N. Cochetkov, "The special theory of relativity and the conservation of momentum," submitted February 18, 2010, General Science Journal, http://www.wbabin.net/papers.htm.

[2] V. N. Cochetkov, "Special relativity fails to conserve momentum," NPA Conference, June 2010 , Long Beach, CA, in this Proceedings.

[3] M. H. Brill, "Herky-Jerky Box: Proof that momentum isn't conserved in special relativity, " *Physics Essays* **23**, 333 (2010).

[4] T. E. Phipps, Jr, *Heretical Verities: Mathematical Themes in Physical Description*. Urbana, IL, Classic Nonfiction Library, 1986, p. 10.

[1] To find the shape, define a parameter r ($0 \le r \le 1$), multiply the sines and cosines by r in Eqs.(1) and (6), solve Eqs. (6) for x′ quantities, and back-substitute to find t and y′ quantities.