Eccentricity Functions in the Higher Degree and Order Sectorial Gravitational Harmonic Coefficients

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In the study of an Earth orbiting satellite, the terms of the series expansion of the Earth's gravitational potential can be expressed as functions of the eccentricity of the satellite. These functions are also known as eccentricity functions. The series expansion of these functions given by Kaula [2] appears to result in instabilities at high eccentricities. When calculating the eccentricity functions, researchers resort to numerical integration techniques instead. The approach followed in this contribution bypasses the problem of instability at high eccentricities by using a Hansen coefficient definition. As a test, we first calculate analytical expressions for various known eccentricity functions and then we proceed with the calculation of the eccentricity functions associated with degree and order 20, 30, 40, 50 sectorial harmonic coefficient expansion of the gravitational potential. Our calculation demonstrates the efficiency of Hansen coefficient approach that differs from that given by Kaula. It is efficient, fast, and can easily be performed with the help of a personal computer, with no instabilities at higher eccentricities.

Key Words: Gravitational field, harmonic coefficients, Hansen coefficients, eccentricity functions.

1. Introduction

The first kind of eccentricity functions related to the satellite theory, appears in the expression of the disturbing potential due to the primary body. These functions are denoted as $G(e)_{\ell,p,q}$. Following Giacaglia [1] we write the eccentricity functions in the following way

$$G(e)_{\ell,p,q} = X_{(\ell-2p+q)}^{-(\ell+1),(\ell-2p)}, \tag{1}$$

where $X_k^{n,m}$ are the so called Hansen coefficients defined as follows

$$X_k^{n,m} = \left(1+b^2\right)^{-(n+1)} \sum_{s=0}^{s'} \sum_{t=0}^{t'} \binom{n-m+1}{s} \binom{n+m+1}{t} (-b)^{s+t} J_{k-m-s+t}(k\,e), \tag{2}$$

where e is the eccentricity of the orbiting satellite, $b = e/(1+\sqrt{1-e^2})$, $J(k\,e)$ is the Bessel function of the first kind of argument ke, and s',t' are the upper limits of the sums. The derivation of the G functions in terms of an infinite series is explained in Kaula [2] and Caputo [3]; their results are based on the works of Cayley [4] and Tisserand [5]. The series of a particular G is identified by the integer indices ℓ,p,q. The first index ℓ is the degree of the spherical harmonic expansion of the disturbing potential. The second index p can be positive or zero, and satisfies the relation $p \le \ell$. The third index q can be negative or positive, and its magnitude determines the power of the first term in the infinite series. Finally, most of the eccentricity functions defined by Eq. (1) exhibit symmetry properties.

The eccentricity functions $G_{\ell,p,q}(e)$ express the satellite eccentricity effect on the harmonic coefficients $C_{\ell m}$ and $S_{\ell m}$ of the disturbing potential. They arise from circular to elliptic transformations of the orbital radius r and the true anomaly f of the satellite to its semimajor axis a, the eccentricity e, and the mean anomaly M.

These orbital elements originally appear in the series expansion of the Earth's gravitational potential. For small eccentricities, we have that $O\left(G(e)_{\ell pq}\right) = e^{|q|}$ [6]. The calculation of eccentricity functions plays a vital role in the Earth and planetary gravity field modeling. In high degree and order gravity models, high order eccentricity functions are calculated. Other authors like Gooding and King-Hele [7] calculate the eccentricity functions using a numerical quadrature of the integral [8]

$$G_{\ell pq}(e) = \frac{1}{\pi} \int_0^{\pi} \frac{\cos\left((\ell-2p)f - (\ell-2p+q)M\right)}{(1-e\cos E)} dE, \tag{3}$$

where E is the eccentric anomaly. The integration is performed using a Newton-Cotes formula of the form:

$$\int_a^b f(x)\,dx \cong \frac{b-a}{90}\left(7f_0 + 32f_1 + 12f_2 + 32f_3 + f_4\right), \tag{4}$$

where $f_i = f\left(a + 0.25i(b-a)\right)$, $i = 0,1,2,3,4$. An advantage of the numerical techniques is that they help avoid instability problems. In particular, the formulation presented in Kaula [2] can result in unstable results at high eccentricities. Although his mathematical formulation is rigorous, their direct evaluation from the series summation becomes inaccurate on a computer of finite precision [8]. It is therefore important to have an effective way of obtaining analytical as well as numerical results for these functions, a way that does not suffer from instabilities and inaccuracies at high eccentricities. The goal of this contribution is to establish, test, and offer an analytical method of evaluating the eccentricity functions that is fast, efficient, and precise. To achieve our goal, we adopt Giacaglia's formulation given in Eq. (1) as a possible way of obtaining the eccentricity functions. Then, with the help of a personal computer, we proceed in testing for the validity, and efficiency of Eq. (1) by calculating various eccentricity functions. In particular, we derive known analytical results for satellite eccentricity functions that are already tabulated in Kaula [2]. A gravitational field model requires the calculation of zonal, sectorial, and tesseral spherical harmonics coefficients. These coefficients are defined as follows: zonal $\ell \neq 0$, $m = 0$, sectorial $\ell = m$ and tesseral $\ell \neq m \neq 0$. To demonstrate our approach, we proceed only with the calculation of higher degree and order eccentricity functions associated with the calculation of sectorial spherical harmonic coefficients. We could have equally chosen any other kind of eccentricity functions associated with the rest of the coefficients involved in the gravitational potential modeling. This choice of indices $\ell = m$ by no means constitutes a limitation in the applicability of Eq. (1), and such index choices will constitute a future contribution. Eccentricity functions of higher degree might exist in gravitational models of higher harmonics in the case of the Earth and other planets (e.g. Mars), but to the best of our knowledge they are not easily accessible. Next, we proceed with the derivation of $\ell = m = 20, 30, 40, 50$ degree and order eccentricity functions that we were not able to find them tabulated in the standard bibliography. Of course, this is only a small number of eccentricity functions when compared to those required in the modeling of the Earth or any planetary gravitational field model. Such a calculation is helpful in satellite gravity field modeling, where fast analytically and numerically computing eccentricity functions values is necessary.

2. Eccentricity Function Calculation

With reference to Lemoine [8], we say that the formulas given by Kaula [2] become computationally unstable if high eccentricity values are used in numerical calculations. First, we test Eq. (1) by verifying the known tabulated eccentricity function calculations. For that, we choose to test Eq. (2) by calculating $G_{2,2,1}(e)$ and $G_{6,6,4}(e)$ respectively. Therefore, using the corresponding Hansen coefficients, Eq (1) gives:

$$G(e)_{2,2,1} = X_{-1}^{-3,-2} \quad (5)$$

$$G(e)_{6,6,4} = X_{-2}^{-7,-6}. \quad (6)$$

After summing Eq. (2) from 0 to 6, and substituting the corresponding index values $n = -3$, $m = -2$, $k = -1$ and $n = -7$, $m = -6$, $k = -2$ as well as the value of the parameter b we obtain up to O(e6):

$$X_{-1}^{-3,-2} = \left(1 + \frac{e^2}{\left(1+\sqrt{1-e^2}\right)^2}\right)^2 \left(\begin{array}{l} -J_1(e) + \dfrac{4eJ_2(e)}{1+\sqrt{1-e^2}} - \dfrac{10e^2 J_3(e)}{\left(1+\sqrt{1-e^2}\right)^2} \\ + \dfrac{204e^3 J_4(e)}{\left(1+\sqrt{1-e^2}\right)^3} - \dfrac{35e^4 J_5(e)}{\left(1+\sqrt{1-e^2}\right)^4} \\ + \dfrac{56e^5 J_6(e)}{\left(1+\sqrt{1-e^2}\right)^5} - \dfrac{84e^6 J_7(e)}{\left(1+\sqrt{1-e^2}\right)^6} - .... \end{array} \right) \quad (7)$$

and

$$X_{-2}^{-7,-6} = \left(1 + \frac{e^2}{\left(1+\sqrt{1-e^2}\right)^2}\right)^6 \left(\begin{array}{l} J_4(2e) - \dfrac{12e J_5(2e)}{1+\sqrt{1-e^2}} + \dfrac{78e^2 J_6(2e)}{\left(1+\sqrt{1-e^2}\right)^2} \\ - \dfrac{364e^3 J_7(2e)}{\left(1+\sqrt{1-e^2}\right)^3} + \dfrac{1365e^4 J_8(2e)}{\left(1+\sqrt{1-e^2}\right)^4} \\ - \dfrac{4368e^5 J_9(2e)}{\left(1+\sqrt{1-e^2}\right)^5} + \dfrac{12376e^6 J_{10}(2e)}{\left(1+\sqrt{1-e^2}\right)^6} - .... \end{array} \right). \quad (8)$$

Next, using Eq. (7) and (8) we obtain a series expansion of order six in eccentricity that involves the Bessel function. Finally, with the help of Mathematica 5.2 for Windows, the series expansion of Eqs. (7) and (8) simplify to:

$$G(e)_{2,2,1} = X_{-1}^{-3,-2} = -\frac{e}{2} + \frac{e^3}{16} + \frac{5e^5}{384} + O(e^7) + ... \quad (9)$$

$$G(e)_{6,6,4} = X_{-2}^{-7,-6} = \frac{e^4}{24} + \frac{e^6}{270} + O(e^8) + ... \quad (10)$$

The calculated eccentricity functions are identical to those given in Chao [9] and also Kaula [2], with the only difference that our results contain terms of higher order in eccentricity i.e., $O(e^5)$ and $O(e^6)$ something that was our choice.

3. Eccentricity Function Results

We calculate and tabulate the eccentricity functions $G(e)_{\ell,p,q}$, associated with the $\ell = m$ = 20, 30, 40, 50 degree and order harmonic coefficient expansion of the gravitational potential using Mathematica 5.2 for Windows. As a demonstration, we obtain the eccentricity functions $G_{40,21,2}(e)$ and $G_{50,25,0}(e)$. This is achieved via the Hansen coefficient expansion that is eventually related to a Bessel function of the first kind. Following identical steps with those indicated in Eq. (5) and (6) we obtain:

$$G_{40,21,2}(e) = X_0^{-41,-2} = \left(1 + \frac{e^2}{\left(1+\sqrt{1-e^2}\right)^2}\right)^{40} \left(\begin{array}{l} \dfrac{741e^2}{\left(1+\sqrt{1-e^2}\right)^2} + \dfrac{414960e^4}{\left(1+\sqrt{1-e^2}\right)^4} \\ + \dfrac{9144681e^6}{\left(1+\sqrt{1-e^2}\right)^6} + \dfrac{11266246992e^8}{\left(1+\sqrt{1-e^2}\right)^8} \\ + \dfrac{908341163730e^{10}}{\left(1+\sqrt{1-e^2}\right)^{10}} + ... \end{array} \right). \quad (11)$$

Next, we expand Eq. (11) in series of eccentricity up to order six and adding the resulting terms, Eq. (11) simplifies to

$$G_{40,21,2}(e) = X_0^{-41,-2} = \frac{741e^2}{4} + \frac{223041e^4}{8} + \frac{110405295e^6}{64} + O(e^8). \quad (12)$$

Similarly, following the same approach we obtain

$$G_{50,25,0} = X_0^{-51,0} =$$

$$\left(1+\frac{e^2}{\left(1+\sqrt{1-e^2}\right)^2}\right)^{50}\left(\begin{array}{l}1+\frac{840401256605625e^{12}}{\left(1+\sqrt{1-e^2}\right)^{12}}+\frac{10001469500100e^{10}}{\left(1+\sqrt{1-e^2}\right)^{10}}\\+\frac{85746480625e^8}{\left(1+\sqrt{1-e^2}\right)^8}+\frac{488410000e^6}{\left(1+\sqrt{1-e^2}\right)^6}\\+\frac{1625625e^4}{\left(1+\sqrt{1-e^2}\right)^4}+\frac{2500e^2}{\left(1+\sqrt{1-e^2}\right)^2}\end{array}\right),(13)$$

which also simplifies to

$$G_{50,25,0}=X_0^{-51,0}=1+\frac{1275e^2}{2}+\frac{878475e^4}{8}+\frac{144948375e^6}{16}. \quad (14)$$

We now proceed with the calculation and tabulation of eccentricity functions associated with $\ell=m=20,\ 30,\ 40$, 50 degree and order harmonic expansion.

$G_{\ell,p,q}$			$X_k^{n,m}$			$G(e)_{\ell,p,q}$
ℓ	p	q	n	m	k	
20	10	0	-21	0	0	1+105e2+26565e4/8+221375e6/6+…
20	11	2	-21	-2	0	171e2/2+14421e4/8+1081575e6/32+…
20	12	4	-21	-4	0	969e4/4+14535e6/2+…
20	13	6	-21	-6	0	6783e6/16+…

Table 1. Eccentricity functions for $\ell=m=20$ using the Hansen coefficient definition.

$G_{\ell,p,q}$			$X_k^{n,m}$			$G(e)_{\ell,p,q}$
ℓ	p	q	n	m	k	
0	12	-6	-31	6	0	118755e6/16+….
30	13	-4	-31	4	0	23751e4/16+2826369e6/32+….
30	14	-2	-31	2	0	203e2/2+8932e4+1328635e6/4+…
30	15	0	-31	0	0	1+465e2/2+15345e4+1014475e6/2+...
30	16	2	-31	-2	0	203e2/2+8932e4+1328635e6/4+…
30	17	4	-31	-4	0	23751e4/16+2826369e6/32+…
30	18	6	-31	-6	0	118775e6/16+…
30	29	28	-31	-28	0	0
30	30	30	-31	-30	0	0

Table 2. Eccentricity functions for $\ell=m=30$ using the Hansen coefficient definition.

$G_{\ell,p,q}$			$X_k^{n,m}$			$G(e)_{\ell,p,q}$
ℓ	p	q	n	m	k	
40	17	-6	-41	6	0	32626231e6/64+….
40	18	-4	-41	4	0	82251e4/16+8142849e6/16+…
40	19	-2	-41	2	0	741e2/4+223041e4/8+110405295e6/68+…
40	20	0	-41	0	0	1+410e2+185115e4/4+10181325e6/4+…
40	21	2	-41	-2	0	741e2/4+223041e4/8+110405295e6/68+…
40	22	4	-41	-4	0	82251e4/16+8142849e6/16+…
40	23	6	-41	-6	0	32626231e6/64+…

Table 3. Eccentricity functions for $\ell=m=40$ using the Hansen coefficient definition.

$G_{\ell,p,q}$			$X_k^{n,m}$			$G(e)_{\ell,p,q}$
ℓ	p	q	n	m	k	
50	22	-6	-51	6	0	1747977e6/8+…
50	23	-4	-51	4	0	59269e4/4+15731793e6/8+…
50	24	-2	-51	2	0	294e2+67522e4+50135085e6/8+…
50	25	0	-51	0	0	1+1275e2/2+878475e4/8+144948375e6/16+…
50	26	2	-51	-2	0	294e2+67522e4+50135085e6/8+…
50	27	4	-51	-4	0	59269e4/4+15731793e6/8+…
50	28	6	-51	-6	0	1747977e6/8+…

Table 4. Eccentricity functions for $\ell=m=50$ using the Hansen coefficient definition.

4. Conclusion

Using Mathematica 5.2 for Windows, we tested Giacaglia's definition of the eccentricity functions. Next using the same definition we calculated various high degree and order eccentricity functions, required in the gravitational field modelling. To ensure the workability of Giacaglia's approach, in this contribution we derived with success two different eccentricity functions. First $G(e)_{2,2,1}$, which is one of those tabulated by Kaula [2], and, then $G(e)_{6,6,4}$ which is one of those tabulated by Chao [9]. After the workability of the method was established, we proceeded with the goal of our paper, which was the calculation of the eccentricity functions associated with higher order sectorial harmonic coefficients. In particular we successfully calculated and tabulated the various eccentricity functions associated with the $\ell = m$ = 20, 30, 40, 50 degree and order harmonic coefficient expansion of the gravitational potential. Since people today use numerical methods in calculating these functions, and taking into account that Kaula's formulas become computationally unstable at high eccentricities, this contribution based on Giacaglia's result offers an alternative, efficient, and fast way of calculating the required functions and at the same time provides analytical results that are stable at high eccentricities.

References

[1] G. E. O. Giacaglia, "A Note on Hansen Coefficients in Satellite Theory", *Celest. Mech.* **14**: 515-523 (1976).

[2] W. Kaula, **Satellite Geodesy** (Dover Publications, 2000).

[3] M. Caputo, **The Gravity Field of the Earth, from Classical and Modern Methods**, International geophysics series, Vol. 10 (New York: Academic Press, 1967).

[4] Cayley (1861)

[5] Tisserand (1960).

[6] K. Lambeck, **Geophysical Geodesy** (Oxford: Charendon Press, 1988).

[7] R. H. Gooding & D. G. King-Hele, "Explicit Forms of Some Functions Arising in the Analysis of Resonant Satellite Orbits", *Proceedings of the Royal Society A* **422**: 241-259 (1989).

[8] F. G. Lemoine, "Mars: The Dynamics of Orbiting Satellites and Gravity Model Development", PhD Thesis (University of Colorado at Boulder, 1992).

[9] G. Chao, **Applied Orbit Perturbation and Maintenance** (The Aerospace Press, American Institute of Aeronautics and Astronautics Inc, Reston, Virginia, 2005).

[10] R. L. Burden & J. D. Faires, **Numerical Analysis** (Boston: Prindle, Weber, and Schmidt, 1985).