# Celestial Mechanics in Spherical Space

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The Dynamic Universe model[(1)] describes space as the surface of a four-dimensional sphere expanding in the direction of the 4-radius. Instead of being defined as a physical constant the velocity of light becomes determined by the velocity of space in the fourth dimension. The changing velocity of light and the dynamics of space allow time to be defined as a universal scalar. Local mass centres modify space in the fourth dimension, giving a space geometry with features that are closely related to those of the Schwarzschild metrics based on four-dimensional space-time. In the modified space geometry the local velocity of light is a function of the local tilting of space in the fourth dimension. The precise geometry of space makes it possible to solve the effect of the 4-D topology on Kepler's laws and the orbital equation. The perihelion shift of planetary orbits can be derived in closed mathematical form as the rotation of the main axis of Kepler's orbit relative to the reference co-ordinate system. For a full revolution the rotation is $Dj = 6p GM/c^2 a(1-e^2)$ like the corresponding prediction in the general theory of relativity .
*Keywords*: Cosmology, zero-energy principle, Dynamic Universe, celestial mechanics, planetary orbits, perihelion shift

# Introduction

In the theory of general relativity (GR), the geometry of homogeneous space is described with a four-dimensional spherically symmetric squared line element

$$ds^2 = -c^2 dt^2 + dr^2 + r^2\left(dq^2 + \sin^2 q\, dj^2\right) \tag{1}$$

where *cdt* is considered as the time-like fourth dimension.

When a central mass is introduced at the origin equation (1) can be expressed in the form of the Schwarzschild metrics as

$$ds^2 = c^2\left(1 - 2GM/rc^2\right)dt^2 + \frac{dr^2}{(1 - 2GM/rc^2)} + r^2\left(dq^2 + \sin^2 q\ dj^2\right) \tag{2}$$

In the first term, factor $(1 - 2GM/rc^2)$ is regarded as the reduction in the flow of time due to the local gravitational centre. In the second term the same factor appears as the increase in the radial line element close to the mass centre. The final term shows the tangential line elements, which are affected by the mass centre through radius *r* as the integrated effect of the modified *dr*. The equation for a planetary orbit is obtained by the solution of geodesic equations derived from the Schwarzschild metrics. The solution for the inverse of distance *r* is[2]

$$W = \frac{1}{r} = \frac{1}{a\left(1-e^2\right)}\left[1 + e\cdot\sin j - \frac{3GMe}{c^2 a\left(1-e^2\right)} j\ \cos j\right] + \Delta W \tag{3}$$

where $2a$ is the length of the major axis and *e* is the eccentricity of the elliptical orbit. Term *DW* is

$$\Delta W = \frac{GM}{c^2 a^2 (1-e^2)^2}\left[e^2 \cos^2 j + 3 + e^2\right] \tag{4}$$

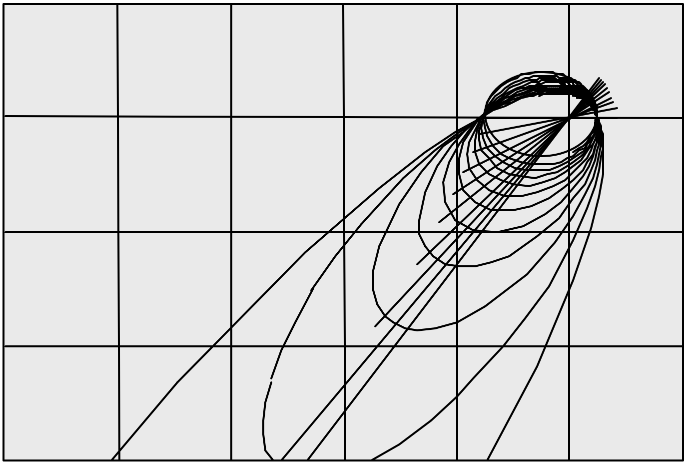

FIG. 1. Development of a planetary orbit as predicted by GR according to equation (3) for $d = GM/ac^2 = 4 \cdot 10^{-3}$ and $e = 0.6$. Beyond twelve revolutions the orbit extends to infi nity. For the first revolution the perihelion shift is $Dj_{2p}(0) = 6pGM/[c^2a(1-e^2)]$. (The gravitational factor of Mercury is $d \approx 2.6 \cdot 10^{-8}$ and the binary pulsar PSR 1913+16 $d \approx 4.6 \cdot 10^{-6}$).

The perihelion advance results from the third term in the square brackets in equation (3), which for the first revolution, with $j = 0$ to $2p$, is

$$\Delta j = \frac{6p\,GM}{c^2 a(1-e^2)} \tag{5}$$

which is consistent with the observed perihelion shifts of Mercury and several binary pulsars. A detailed analysis of equation (3), however, shows that the orbit predicted for multiple revolutions ($j = n \cdot 2p$) exhibits a decreasing perihelion advance and is associated with a cumulative increase of eccentricity (see FIG. 1). As clearly illustrated by orbits with high eccentricity and high gravitational factor, equation (3) leads to an increasing asymmetric orbit with a diminishing perihelion rotation. The orbit calculation does not take into account possible effects of the gravitational radiation predicted by GR.

The instability of the orbit predicted by equation (3) is inconsistent with observation, which indicates a problem either in the Schwarzschild metrics or in approximations made in solving the geodesic equations.

The Dynamic Universe model[(1)] describes three-dimensional space as a spherically symmetric structure closed through the fourth dimension. As a consequence of the balance of the energies of motion and gravitation in the structure space, the surface of the expanding four-dimensional sphere is in motion along the radius in the fourth dimension. The velocity of the motion in the fourth dimension appears as the maximum velocity obtainable in space. As a result of the conservation of the total energy, space is tilted in the fourth dimension near mass centres, which makes the direction of the local fourth dimension deviate from the direction of the fourth dimension in non-tilted, apparent homogeneous space. In tilted space the line element can be expressed as

$$ds^2 = c_{0d}^2 \left(1-\mathbf{d}\right)^2 dt^2 + \frac{dr_{0d}^2}{\left(1-\mathbf{d}\right)^2} + r_{0d}^2 \left(d\mathbf{q}^{\,2} + \sin^2\mathbf{q}\; d\mathbf{j}^{\,2}\right) \qquad (6)$$

where $\mathbf{d} = GM/rc^2$ is the local gravitational factor, $c_{0d}$ is the imaginary velocity of space in non-tilted space, $r_{0d}$ the line element in the direction of the non-tilted space and $f$ the tilting angle at gravitational state $\mathbf{d}$ ,$\cos f = (1-\mathbf{d})$ .

The first term in equation (6) describes the motion of space and in the direction of the local fourth dimension showing the effect of the reduction of the local velocity of light due to the tilting of space near a mass centre. The tilting of space also results in a lengthening of the line element *dr* in the radial direction (towards the mass centre). The

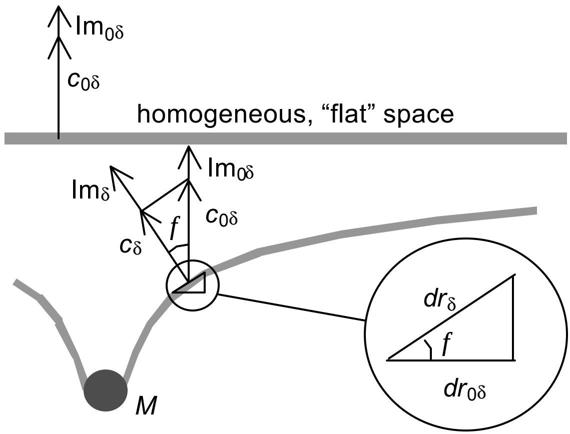


FIG. 2. In the DU model space is a dynamic structure moving at velocity $c_{0d}$ in the direction of the fourth dimension. In the vicinity of a mass centre the fourth dimension of local space is tilted by angle $f$ relative to apparent homogeneous space. The velocity of space in the local fourth dimension, $c_d$, is $c_d = c_{0d}\cos f$ and the length of the local radial line element $dr_d$ is $dr_d = dr_{0d}/\cos f$ where distance $dr_{0d}$ is measured in the direction of apparent homogeneous space.

second term shows the effect of tilting on the line element $dr$ [$dr_d = dr_{0d}/\cos f$]. In contrast to the Schwarzschild metrics, the fourth dimension, in the DU, is purely geometrical in nature. Other differences to the Schwarzschild metrics are that the line element $dr_{0d}$ in the second term of equation (6) and distance $r_{0d}$ in the last term are measured in the direction of non-tilted, apparent homogeneous space (see FIG. 2).

The Dynamic Universe model allows the solution of planetary orbits in closed mathematical form following the procedure used in deriving the Kepler's orbital equations. The solution is first derived as

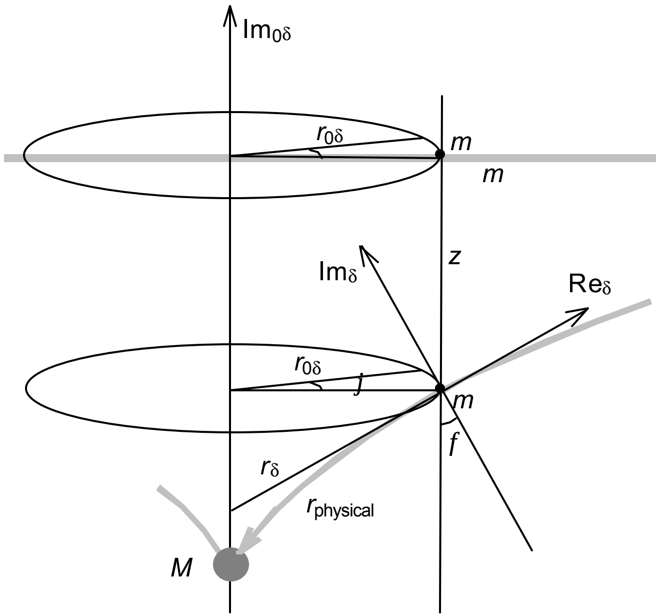


FIG. 3. The cylinder co-ordinate system used in orbital calculations. The local complex co-ordinate system, $\mathrm{Im}_d$–$\mathrm{Re}_d$ at object *m*, is also illustrated. The radial line element $dr_{0d}$ in the DU metrics has the direction of $r_{0d}$.

a plane projection of the obit on a plane in the direction of apparent homogeneous space, "flat space" (see FIG. 3). The radial acceleration needed in the derivation is the flat space component of the acceleration of free fall in locally tilted space. The acceleration of free fall in locally tilted space, like the curvature itself are derived from the conservation of total energy.

A complete orbital equation is obtained by adding the *z*-co-ordinate, the distance from a selected reference plane in the direction of apparent homogeneous space, to the planar solution.

# Acceleration in locally curved space

Kepler's laws are based on Newtonian mechanics in the orbital plane. In Newtonian mechanics the equation of motion for mass $m$ in the local gravitational frame around mass $M$ is expressed as

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -\frac{m\vec{r}}{r^3} \tag{7}$$

where $m$ is obtained as

$$m = G(M+m) \tag{8}$$

when the gravitational constant is combined with the effect of the two masses. Equation (7) states the connection between radial acceleration and the Newtonian gravitational force.

According to the DU model, the orthogonal sum of the velocities of free fall $v_{ff}$ and the local imaginary velocity of space $c_d$ is equal to the imaginary velocity of apparent homogeneous space $c_{0d}$ (see FIG.2). Accordingly, $v_{ff}$ can be expressed in terms of $c_{0d}$ and the gravitational factor $d$ as

$$v_{ff} = \sqrt{c_{0d}^2 - c_d^2} = c_{0d}\sqrt{1-(1-d)^2} \tag{9}$$

The acceleration of free fall can be expressed as

$$a_{ff} = \frac{dv_{ff}}{dt} = \frac{dv_{ff}}{dr_{0d}} \cdot \frac{dr_{0d}}{dt} = \frac{dv_{ff}}{dr_{0d}} v_{ff(0d)} = \frac{dv_{ff}}{dr_{0d}} v_{ff}(1-d) \tag{10}$$

where $v_{ff(0d)}$ is the component velocity $v_{ff}$ in the direction of $r_{0d}$, $v_{ff(0d)} = v_{ff}(1-d)$.

As solved from (10) acceleration $a_{ff}$ in the direction of local space is

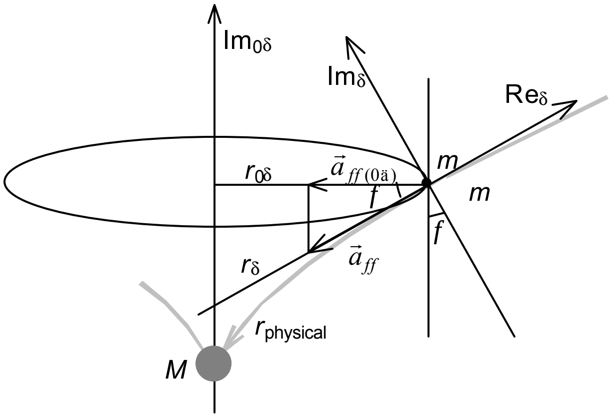


FIG. 4. Acceleration $\vec{a}_{ff(0d)}$ is the flat space component of acceleration $\vec{a}_{ff}$ . Acceleration $\vec{a}_{ff(0d)}$ has the direction of $r_{0d}$ .

$$a_{ff} = \frac{GM}{r_{0d}^2}(1-d)^2 \tag{11}$$

and the component $a_{ff(0d)}$ in the direction of flat space

$$a_{ff(0d)} = a_{ff}(1-d) = \frac{GM}{r_{0d}^2}(1-d)^3 \tag{12}$$

(see FIG. 4).

Applying acceleration $\vec{a}_{ff(0d)}$ in equation (12) for $\vec{a}$ in equation (7), the DU flat space substitute of the Newtonian equation (7) can be expressed as

$$\vec{a}_{ff(0d)} = \frac{d^2\vec{r}_{0d}}{dt^2} = -m'\frac{(1-d)^3}{r_{0d}^3}\ \vec{r}_{0d} \tag{13}$$

By expressing the gravitational factor $d$ in terms of the critical radius $r_c$,

$$d = \frac{r_c}{r_{0d}} \quad \text{where} \quad r_c \equiv \frac{m'}{c^2} \tag{14}$$

equation (13) can be expressed in form

$$\frac{d^2\vec{r}_{0d}}{dt^2} = -m'\frac{(1 - r_c / r_{0d})^3}{r_{0d}^3}\ \vec{r}_{0d} \approx -m'\left(1 - \frac{3r_c}{r_{0d}}\right)\frac{\vec{r}_{0d}}{r_{0d}^3} \tag{15}$$

Equation (15) is the equation of motion to be used in the derivation of the orbital equation on the base plane in the direction of apparent homogeneous space.

## The eccentricity vector

The angular momentum per mass unit (related to the orbital velocity in the direction of the base plane) is denoted as

$$\vec{k}_{0d} = \vec{r}_{0d} \times \dot{\vec{r}}_{0d} \tag{16}$$

The time derivative of $\vec{k}_{0d}$ is

$$\dot{\vec{k}}_{0d} = \vec{r}_{0d} \times \ddot{\vec{r}}_{0d} + \dot{\vec{r}}_{0d} \times \dot{\vec{r}}_{0d} = \vec{r}_{0d} \times \ddot{\vec{r}}_{0d} \tag{17}$$

which, in order to conserve angular momentum, must be equal to zero. Substituting (15) for $\ddot{\vec{r}}_{0d}$ in (17) we get

$$\dot{\vec{k}}_{0d} = \vec{r}_{0d} \times \vec{r}_{0d}\ \frac{-m'(1 - 3r_c / r_{0d})}{r_{0d}^3} = 0 \tag{18}$$

To determine vector $\vec{e}_{0d}$ we form the vector product $\vec{k}_{0d} \times \ddot{\vec{r}}_{0d}$

$$\vec{k}_{0d} \times \ddot{\vec{r}}_{0d} = (\vec{r}_{0d} \times \dot{\vec{r}}_{0d}) \times \vec{r}_{0d} \frac{-m'(1-3r_c/r_{0d})}{r_{0d}^3} \tag{19}$$

which can be expressed in the form

$$\vec{k}_{0d} \times \ddot{\vec{r}}_{0d} = \frac{-m'(1-3r_c/r_{0d})}{r_{0d}^3} \left[ (\vec{r}_{0d} \cdot \vec{r}_{0d}) \dot{\vec{r}}_{0d} - (\vec{r}_{0d} \cdot \dot{\vec{r}}_{0d}) \vec{r}_{0d} \right] \tag{20}$$

Given that the time derivative of distance $r_{0d}$, $\dot{r}_{0d}$, is the component of $\dot{\vec{r}}_{0d}$ in the direction of $\vec{r}_{0d}$, it is possible to express the scalar time derivative $\dot{r}_{0d}$ in form of the point product

$$\dot{r}_{0d} = \frac{\vec{r}_{0d}}{r_{0d}} \cdot \dot{\vec{r}}_{0d} \tag{21}$$

and, accordingly, the point product in the second term in the square brackets in equation (20) can be expressed as

$$\vec{r}_{0d} \cdot \dot{\vec{r}}_{0d} = r_{0d} \quad \dot{r}_{0d} \tag{22}$$

Equation (20) can now be expressed as

$$\vec{k}_{0d} \times \ddot{\vec{r}}_{0d} = -m'(1-3r_c/r_{0d}) \left[ \frac{1}{r_{0d}} \dot{\vec{r}}_{0d} - \frac{\dot{r}_{0d}}{r_{0d}^2} \vec{r}_{0d} \right] \tag{23}$$

where the expression in parenthesis can be identified as the time derivative

$$\left[ \frac{\dot{\vec{r}}_{0d}}{r_{0d}} - \frac{\dot{r}_{0d}}{r_{0d}^2} \vec{r}_{0d} \right] = \frac{d(\vec{r}_{0d}/r_{0d})}{dt} \tag{24}$$

Equation (23) can now be expressed as

$$\vec{k}_{0d} \times \ddot{\vec{r}}_{0d} = -m' \frac{d(\vec{r}_{0d}/r_{0d})}{dt} + \frac{3m' r_c}{r_{0d}} \frac{d(\vec{r}_{0d}/r_{0d})}{dt} \tag{25}$$

As shown in equation (18) $\dot{\vec{k}}_{0d}$ , is zero, which means that

$$\vec{k}_{0d} \times \ddot{\vec{r}}_{0d} = \frac{d(\vec{k}_{0d} \times \dot{\vec{r}}_{0d})}{dt} \tag{26}$$

Combining equations (25) and (26) gives

$$\frac{d(\vec{k}_{0d} \times \dot{\vec{r}}_{0d})}{dt} + m' \frac{d(\vec{r}_{0d} / r_{0d})}{dt} = \frac{3m' r_c}{r_{0d}} \frac{d(\vec{r}_{0d} / r_{0d})}{dt} \tag{27}$$

which can be written in the form

$$\frac{d\left[(\vec{k}_{0d} \times \dot{\vec{r}}_{0d}) + m' \vec{r}_{0d} / r_{0d}\right]}{dt} = \frac{3m' r_c}{r_{0d}} \frac{d(\vec{r}_{0d} / r_{0d})}{dt} \tag{28}$$

The expression in parenthesis on the left hand side of the equation is equal to the eccentricity vector $-\vec{e}_{0d} m'$ showing the direction of the perihelion or periastron radius in Kepler's orbital equation. Applying $-\vec{e}_{0d} m'$ in equation (28) we get the time derivative

$$\frac{d\vec{e}_{0d}}{dt} = -\frac{3r_c}{r_{0d}} \frac{d(\vec{r}_{0d} / r_{0d})}{dt} \tag{29}$$

which in Newtonian mechanics is equal to zero. Equation (29) indicates that the eccentricity vector $\vec{e}_{0d}$ changes with time. By solving the derivative of the product in (29) we get

$$\frac{d\vec{e}_{0d}}{dt} = -\frac{3r_c}{r_{0d}} \left[ \frac{1}{r_{0d}} \frac{d\vec{r}_{0d}}{dt} - \frac{\vec{r}_{0d}}{r_{0d}^2} \frac{dr_{0d}}{dt} \right] \tag{30}$$

where $dr_{0d}$ is the differential of the length of radius $r_{0d}$. In polar coordinates on the flat space plane, vector $d\vec{r}_{0d}$ can be expressed as

$$d\vec{r}_{0d} = r_{0d} d j \quad \hat{r}_{(\perp)} + dr_{0d} \quad \hat{r}_{(\|)} \tag{31}$$

where $\hat{r}_{(\perp)}$ and $\hat{r}_{(\|)}$ are the unit vectors perpendicular to $\vec{r}_{0d}$ and in the direction of $\vec{r}_{0d}$, respectively. Substituting (36) into (35) and applying $\hat{r}_{(\|)} = \vec{r}_{0d} / r_{0d}$ gives

$$\frac{d\vec{e}_{0d}}{dt} = -\frac{3r_c}{r_{0d}}\left[\frac{dj}{dt}\ \hat{r}_{(\perp)} + \frac{1}{r_{0d}}\frac{dr_{0d}}{dt}\ \hat{r}_{(\|)} - \frac{1}{r_{0d}}\frac{dr_{0d}}{dt}\ \hat{r}_{(\|)}\right] = -\frac{3r_c}{r_{0d}}\frac{dj}{dt}\ \hat{r}_{(\perp)} \quad (32)$$

As shown by (32), the change of $\vec{e}_{0d}$ occurs as rotational change only, which means that the orbit conserves its eccentricity but is subject to a rotation of the main axis. The condition $d\vec{e}_{0d} / dt = 0$ required by Kepler's orbital equation is achieved in a co-ordinate system with rotation $dy/dt$,

$$\frac{d\vec{e}_{0d}}{dt} + \frac{dy}{dt} = -\frac{3r_c}{r_{0d}}\frac{dj}{dt} + \frac{dy}{dt} = 0 \quad (33)$$

where the rotational velocity of the co-ordinate system relevant with the Kepler's solution is

$$dy = \frac{3r_c}{r_{0d}} dj \quad (34)$$

relative to the reference co-ordinate system at rest.

Applying Kepler's equation

$$r_{0d} = \frac{a(1-e^2)}{(1+e\cdot\cos j)} \quad (35)$$

for $r_{0d}$ in (34) we can express $dy_{0d}$ as

$$dy = \frac{3r_c(1+e\cdot\cos j)}{a(1-e^2)} dj \quad (36)$$

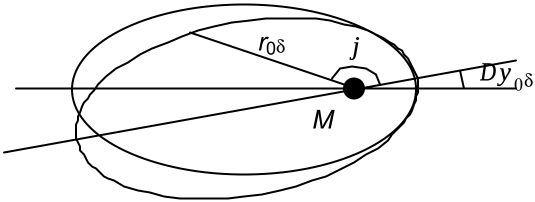


FIG. 5. Perihelion advance results in a rotation of the main axis by $Dy(2p)$ in each revolution.

The rotation of the co-ordinate plane relative to the non-rotating reference co-ordinate plane can now be obtained by integrating (36) as

$$\Delta y = \frac{3r_c(1+e\cdot\cos j)}{a(1-e^2)}\int_0^j (1+e\cdot\cos j)\, dj = \frac{3r_c(j+e\cdot\sin j)}{a(1-e^2)} \quad (37)$$

In the non-rotating reference co-ordinate system, relative to the reference axis at $y_{0d}(0) = 0$, the orbital equation can now be expressed as

$$r_{0d} = \frac{a(1-e^2)}{\left[1+e\cdot\cos(j-\Delta y)\right]} \quad (38)$$

which is the Kepler's equation with a perihelion advance by angle $Dy(j)$.

Setting $j = 2p$ and substituting (14) for $r_c$ in (37), the perihelion advance for a full revolution can be expressed as

$$\Delta y(2p) = \frac{6pm'}{ac^2(1-e^2)} \quad (39)$$

which is the same result as given by the theory of general relativity for perihelion advance as the first approximation in equation (3) (see FIG. 5).

## Kepler's energy integral

To complete the analysis of the orbit on the base plane we now study the energy integral derived from the point product of the velocity and the acceleration:

$$\dot{\vec{r}}_{0d}\cdot\ddot{\vec{r}}_{0d}=\dot{\vec{r}}_{0d}\cdot\frac{-\vec{r}_{0d}\,m'(1-3r_c/r_{0d})}{r_{0d}^3}$$
$$=\frac{-r_{0d}\,m'(1-3r_c/r_{0d})}{r_{0d}^3}\dot{r}_{0d}=\frac{-m'}{r_{0d}^2}\frac{dr_{0d}}{dt}+\frac{3m'r_c}{r_{0d}^3}v_{r(0d)} \quad (40)$$

where $v_{r(0d)}=dr_{0d}/dt$ means the radial velocity on the flat space plane. The first term in the last form of equation (40) can be written in the form

$$\frac{-m'}{r_{0d}^2}\dot{r}_{0d}=\frac{d(m'/r_{0d})}{dt} \quad (41)$$

Substituting (41) into (40) we can write

$$\dot{\vec{r}}_{0d}\cdot\ddot{\vec{r}}_{0d}=\frac{d(m'/r_{0d})}{dt}+\frac{3m'r_c}{r_{0d}^3}\dot{r}_{0d} \quad (42)$$

The point product of the velocity and the acceleration can also be expressed as

$$\dot{\vec{r}}_{0d}\cdot\ddot{\vec{r}}_{0d}=\frac{d(1/2\ \dot{\vec{r}}_{0d}\cdot\dot{\vec{r}}_{0d})}{dt}=\frac{d(v_{r(0d)}^2/2)}{dt} \quad (43)$$

Combining equations (42) and (43) gives

$$\frac{d(v_{r(0d)}^2/2-m'/r_{0d})}{dt}=\dot{h}=\frac{3m'r_c}{r_{0d}^3}\dot{r}_{0d} \tag{44}$$

In Kepler's formalism, the expression in parenthesis on the right hand side is referred to as the energy integral $h$. In the case of Newtonian mechanics the time derivative of $h$ is zero, which means that the sum of the Newtonian kinetic and gravitational energies is conserved.

In Kepler's orbital equation

$$r_{0d}=\frac{k^2/m'}{(1+e\cdot\cos j)}=\frac{a(1-e^2)}{(1+e\cdot\cos j)} \tag{45}$$

constants $m$, $e$, $h$, and $k$ are related as

$$h=\frac{-m'^2(1-e^2)}{2k^2} \quad \text{and} \quad k^2=\frac{-m'^2(1-e^2)}{2h} \tag{45}$$

In order to see the effect of the time-dependent energy integral $h$ in the orbital equation we solve for the time derivative of $k^2$

$$\frac{dk^2}{dt}=\frac{-m'^2(1-e^2)}{2}\frac{(-1)}{h^2}\dot{h}=\frac{m'^2(1-e^2)}{2h}\frac{\dot{h}}{h}=-\frac{k^2}{h}\dot{h} \tag{46}$$

Substituting (44) for $dh/dt$ in (46) gives

$$\frac{dk^2}{dt}=-\frac{k^2}{h}\frac{3m'r_c}{r_{0d}^3}\dot{r}_{0d} \tag{47}$$

and substituting (45) for $h$ in (47) we get

$$\frac{dk^2}{dt}=-\frac{6k^4r_c}{m'r_{0d}^3(1-e^2)}\dot{r}_{0d} \tag{48}$$

By substituting (45) for $r_{0d}$ in (48) we get

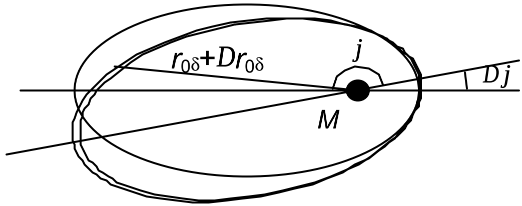


FIG. 6. Kepler's orbit perturbated by distance $Dr_{0d}$ given in equation (53).

$$\frac{dk^2}{dt} = -\frac{6m'^2\, r_c (1+e\cdot\cos j)^3}{k^2(1-e^2)}\;\dot{r}_{0d} \tag{49}$$

and by substituting the time derivative of $r_{0d}$ obtained from equation (45) in (49) we get

$$\frac{dk^2}{dt} = \frac{6m'(1+e\cdot\cos j)}{(1-e^2)} r_c e\cdot\sin j\;\;\frac{dj}{dt} \tag{50}$$

Further applying equation (45) for the differential increase of $r_{0d}$ gives us

$$dr_{0d} = \frac{dk^2}{m'(1+e\cdot\cos j)} \tag{51}$$

By applying $dk^2$, obtained by multiplying (50) by $dt$, to (51) we can express the differential of $dr_{0d}$ in terms of the differential of $dj$ as

$$dr_{0d} = \frac{6r_c e\cdot\sin j}{(1-e^2)}\;dj \tag{52}$$

and the total increase (see FIG. 6) as

$$\Delta r_{0d}(j) = \frac{6r_c e}{(1-e^2)}\int_0^j \sin j\;\;dj = \frac{6r_c e(1-\cos j)}{(1-e^2)} \tag{53}$$

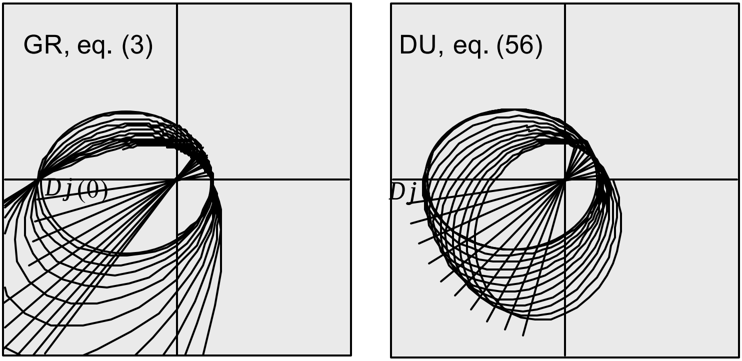


FIG. 7. The development of planetary orbits for the first ten revolutions according to the DU and GR for $d = 4 \cdot 10^{-3}$ and $e = 0.6$. For the first revolution in the GR solution, according to equation (3), and for all revolutions in the DU solution the perihelion shift is $\Delta y\,(2p) = 6pm'/ac^2(1-e^2)$. In the DU solution the perihelion shift stays constant during the rotation of the main axis and the slightly deformed elliptic orbit conserves its shape. In the GR solution, according to equation (3), the shape of the orbit changes due to a reduction of the radius in the 1st and 2nd quarters and an increase of the radius in the 3rd and 4th quarters in the polar co-ordinate system. The deformation of the orbit, as predicted by equation (3), is accompanied by a degrading perihelion shift.

The increase of $r_{0d}$, $Dr_{0d}$, is zero at perihelion and achieves its maximum value at aphelion

$$\text{perihelion: } \Delta r_{0d}(0) = 0 \tag{54}$$

$$\text{aphelion: } \Delta r_{0d}(p) = \frac{12 r_c e}{(1-e^2)} \tag{55}$$

Combining equations (38) and (53) gives the complete orbital equation of the flat space projection of the orbit:

$$r_{0d} = \frac{a(1-e^2)}{[1+e\cdot\cos(j-\Delta y)]} + \frac{6 r_c e[1-\cos(j-\Delta j)]}{(1-e^2)} \tag{56}$$

where, as expressed in equation (37), the perihelion advance $Dj$ is

$$\Delta y = \frac{3r_c(j + e\cdot \sin j)}{a(1-e^2)} \tag{57}$$

Equation (56) is applicable in gravitational potentials $d << 1$ where the approximation $(1-d)^3 \approx (1-3d)$ is sufficiently accurate.

For stable mass centres, the DU orbit conserves its shape, size, and perihelion advance. FIG. 7 compares the developments of the orbits according to equations (56) and (3) corresponding to the DU and GR predictions, respectively.

## The fourth dimension

The orbital co-ordinates are completed by adding the co-ordinate $z$ which extends the orbital calculation made as the base plane projection to actual space curved in the fourth dimension. With reference to [1], the co-ordinate $z$, the distance from the base plane (in the direction of apparent homogeneous space) crossing the orbiting surface at $j = \pm p/2$ can be expressed as

$$z(r_{0d}) = 2\sqrt{2r_c}\left[\sqrt{r_{0d}} - \sqrt{a_{0d}(1-e_{0d}^2)}\right] \tag{58}$$

where $r_{0d}$ is the flat space distance from the centre of the gravitational frame given in equation (56). Expression $a_{0d}(1-e_{0d}^2)$ in equation (58) is the value of $r_{0d}$ at $j_{0d} = p/2$, which is used as the reference value for the $z$-co-ordinate. Equations (56) and (58) give the 4-dimensional co-ordinates of an orbiting object as the function of angle $j_{0d}$ determined relative to the perihelion direction in the flat space projection of the orbit (see FIG. 8).

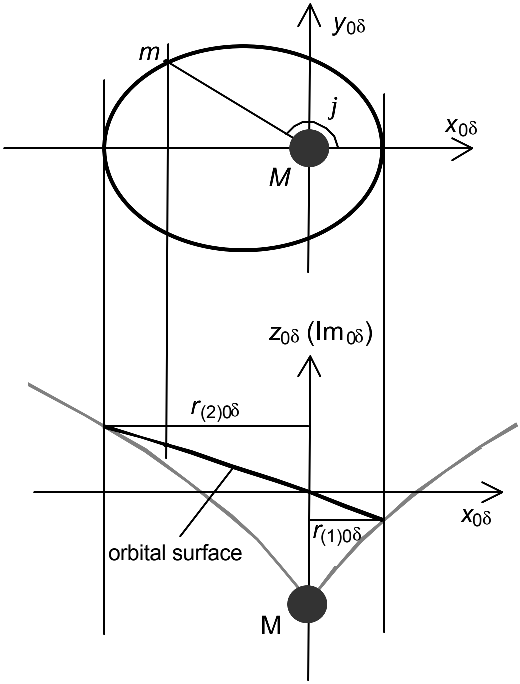


FIG. 8. Projections of an elliptical orbit on the $x_{0d}$–$y_{0d}$ and $x_{0d}$–$y_{0d}$ planes in a gravitational frame around mass centre *M*.

The differential of a line element in the $z_{0d}$-direction can be expressed in terms of the differential in the $r_{0d}$-direction as

$$dz_{0d} = dR''_{0d} = \tan f \;\; dr_{0d} = B dr_{0d} \tag{59}$$

where

$$B = \tan f = \sqrt{\frac{1-(1-d)^2}{(1-d)}} \tag{60}$$

where $\mathbf{d} = r_c / r_{0d} = m' / r_{0d} c^2$ and $dr_{0d}$ can be solved from equations (56) as

$$dr_{0d} = Ad\,\mathbf{j}_{0d} \tag{61}$$

where

$$A = \frac{ae(1-e^2)\cdot\sin(j - \Delta y)}{\left[1+e\cdot\cos(j - \Delta y)\right]^2} + \frac{6r_c e\sin(j - \Delta y)}{(1-e^2)} \tag{62}$$

The total differential of the path can be expressed in cylindrical coordinates as

$$d\hat{s} = dr_{0d}\hat{u}_{r(0d)} + r_{0d}d\,\mathbf{j}_{0d}\hat{u}_{j(0d)} + dz_{0d}\hat{u}_{z(0d)} \tag{63}$$

where $\hat{u}_{r(0d)}, \hat{u}_{j(0d)}, and\ \hat{u}_{z(0d)}$ are unit vectors in the radial and tangential direction of the orbit on the base plane and in the $z_{0d}$-direction.

The squared line element $ds^2$ of an orbit around a mass centre can now be expressed as

$$ds^2 = (r_{0d}^2 + A^2 + A^2B^2)d\,\mathbf{j}_{0d}^2 \tag{64}$$

and the scalar value of the line element as

$$ds = \sqrt{r_{0d}^2 + A^2(1+B^2)}\ \ d\mathbf{j}_{0d} \tag{65}$$

The length of the path along the orbit from $j1$ to $j2$ can be obtained by integrating (65) as

$$s = \int_{j2}^{j1} \sqrt{r_{0d}^2 + A^2(1+B^2)}\ \ d\mathbf{j}_{0d} \tag{66}$$

FIG. 9 shows the $x_{0d}$–$z_{0d}$ profile of the orbit of Mercury in the solar gravitational frame.

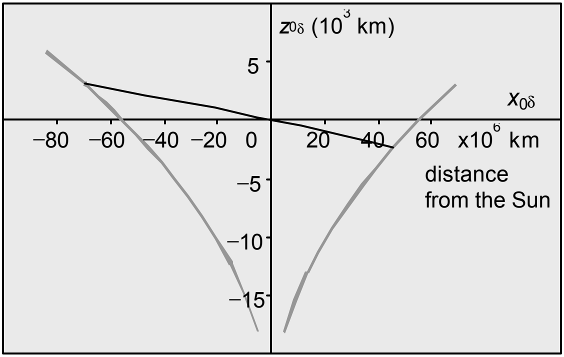


FIG. 9. The $z_{0d}$–$x_{0d}$ profile of the orbit of the planet Mercury. Note the different scales in the $z_{0d}$- and $x_{0d}$- directions.

## Conclusions

The DU model predicts a perihelion shift equal to the shift predicted by the theory of general relativity for the first orbital revolution. A planetary orbit including a *z*-co-ordinate in the fourth dimension, the perihelion shift, and the perturbation of the radial distance can be presented in closed mathematical form in a cylindrical co-ordinate system with the base-plane lying in the direction of the apparent homogeneous space of the gravitational frame of interest.

## Acknowledgements

I am grateful to Dr. Ari Lehto and Dr. Heikki Sipilä for their insightful comments in discussions related to this study. I would also like to thank Professor Raimo Lehti and Professor Matts Roos for encouraging me to carry out the analysis.

## References

[1] Suntola, T., *The Dynamic Universe: A New Perspective on Space and Relativity*, Suntola Consulting Ltd, Oy., Espoo, Finland, 2001, www.sci.fi/~suntola.

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