What Can You Learn From a Hole in the Ground?

. . . and a tent pole, two pieces of rope, and a couple of tent pegs: An Explanation of Tensile-Integrity Structures

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A large class of tensile-integrity structures, popularly called tensegrities, can be derived from (not necessarily regular) **n** sided right prisms by rotating the tops and bottoms clockwise or counter clockwise in relationship to each other. Thus these structures are chiral and exist in both right and left handed mirror image versions of each other. The resulting structures exhibit invariant plan views when parallel projected from the "top" down or "bottom" up along their main axis of rotational symmetry. These plan views are invariant in the same sense that similar triangles are invariant in shape in that they have the same shape with constant angular values no matter what the overall size of the projected structure is and no matter what its overall height to width ratio happens to be. Also, the plan views are identical for both the right and left handed versions. This leads to something called the constant twist angle theorem. This theorem has been demonstrated using calculus, see for instance the Appendix to Chapter 1 of Kenner, Geodesic Math and How to Use It [1]. However, using descriptive geometry and intersecting planes, there is a much simpler and more intuitively obvious way of demonstrating this constant twist angle theorem which bypasses calculus entirely. This leads to a particularly elegant and simple way of drawing a series of plan views of these structures that depends solely on the number of sticks within a structure. This drawing algorithm requires only the tools of Euclidean geometry, namely, unmarked ruler and compass, although in a some cases a graduated protractor is required, say when the circumference of a circle is to be divided into seven equal parts. The first drawing in this series of plan views bears a striking resemblance to the first figure in the first book of Euclid's Elements [4].

1. Introduction

This circle drawing algorithm is a direct consequence of intersecting "planes of force" which can be found in all tensegrities. Every stick stands at the intersection of two intersecting planes of force. Intersecting planes of force are the sole determinant of the overall form of these structures and probably of all tensegrities of whatever class or family whether simple or compound. This suggests a strong connection with folded surface structures and origami.

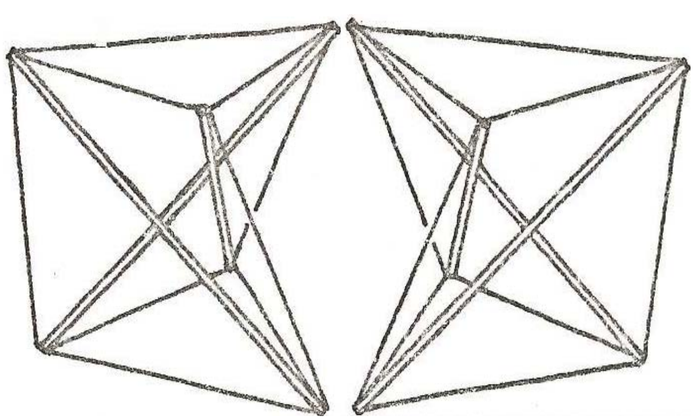


Fig. 1. Left and right handed three stick tensegrities

Tensegrity geometry is still very much an experimental and empirical subject dependent on building physical models to reveal underlying facts and principles. It is not yet a finished and idealized mathematical system of theorems logically derived from a minimum, well established and widely accepted "fixed" set of axioms as is Euclidean plane geometry for instance. But, despite the fact that from an engineering statics point of view tensegrities are non-linear and global in their behavior, a lot of well established classical mathematics, especially projective and incidence geometry as well as elementary Euclidean plane and three dimensional geometry, is applicable. In fact, as an engineering subject you can trace the analysis of the principles underlying tensegrities back to the 1860s and the work of Rankine and Maxwell on the graphical static analysis of bridge frames and trusses [5]. That makes the study at least 150 years old.

Pure tensegrities are structures composed of only discontinuous compressiom and continous tension elements. Compression elements do not touch each other directly but are interconnected solely by tension elements usually, but not necessarily, connected between their end points. The sticks in a tensegrity float in space without touching each other. To put it somewhat paradoxically, the continuous network of tension elements, the "strings", are held apart by being *pushed outward* by the compression elements, the "sticks", while the compression elements are held together by being *pulled inward* by the tension elements. The two sorts of elements work in opposition to each other. It's push against pull. With lines of tension "properly" arranged and tightened between their end points the compression elements cannot and do not touch each other. They can only touch each other if the tension lines are relaxed and become slack. It is my hope to provide some intuitive insight into how it is possible to make sticks float in space and to elucidate the forms these tensegrity arrangements necessarily take. It is my claim that these arrangements follow very simple and strict geometric rules and axioms which combine both metric or distance AND incidence or projective geometry along with balanced and opposed pushing and pulling forces, interconnected vectors so to speak. I hope to show that all tensegrities can be analyzed in terms of intersecting "planes of force". They are not just arbitrary tangles of sticks and strings.

In the first sentence of a recent paper on the finite element analysis of tensegrities the author, a professional structural engineer, states, "It is possible to erect a mast or stick using three cables; more are not needed, *fewer is not possible*." (Emphasis mine.) Based on our everyday commonsense experience this is obviously true; *but* based on experiments that anyone can perform on a desktop or in a back yard, it is definitely false. The minimum number of cables required to hold one end of a mast in an upright position is two, not three, if a certain very reasonable but normally unexamined assumption is changed. How can this be?

Imagine a pole, say 8 feet high. Dig a hole in the ground 4 feet deep and wide enough so that the pole when vertical and standing in the center of the hole is not touching the sides of the hole. The pole should be able to pivot freely back and forth on the bottom of the hole and should stick up above the ground about 4 feet. Tie a rope to the top end of the pole with approximately 5 foot long segments of rope stretched on either side of the pole and peg them to the ground on opposite sides of the pole, say, 3 feet out from each side. Pull the ropes tight as you peg them to the ground. To make things easy we use 3:4:5 Pythagorean triangles opposite each other on opposite sides of the pole so we will automatically have right angles although right angles are not required. It will be found that when pulled tight the two segments of rope and the pole will be pulled into a plane vertical to the ground and the pole will stand upright in the hole. This will be true as long as the ropes are held tight and the pivot point of the pole lies **below** the hinge line or axis of rotation of the two "anchor points" where the rope segments are pegged to the ground. If the anchor points of the rope segments and the pivot point of the pole were all to lie on a common ground plane, as would ordinarily be the case, then they would lie on the same straight line and that straight line would form a hinge line. Then the entire planar structure could pivot around that hinge thus falling flat on the ground, kerplunck! That's why we normally use at least three lines as guy wires to support a pole.

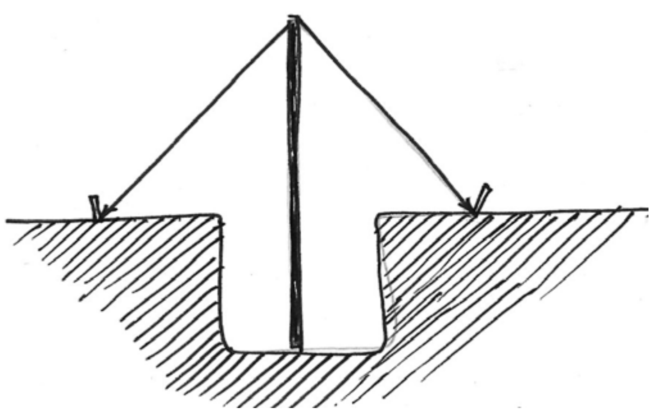


Fig. 2. Side view of a stick standing in a hole in the ground

You can convince yourself of this fact by performing a much simpler experiment, no shovel required. Tie the middle of a string to the end of a stick, say 12 inches long. Put the free end of the stick on your desk top and pull the two string ends tight with your two hands, one holding each string end while keeping your hands above the desktop. (Obviously you won't be able to put them through and below the desktop.) Pull the strings tight into an approximately isosceles triangle. As long as the stick's pivot point on the desk top doesn't slip, you will be able to hold the stick in place by pulling the two strings tight. In fact, you will be able to move the stick around quite a bit while firmly pressing its pivot point to the desk top. Try it and see. A pencil with a rubber eraser at one end makes a good stick with strings tied around the exposed lead end. The rubber eraser helps to keep the pivot point from slipping on the desk.

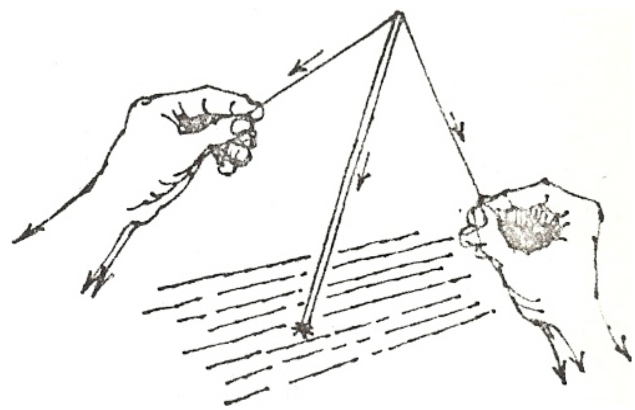


Fig. 3. Two hands, two strings and a stick.

Also, consider holding a bow and arrow with the bow string drawn back and an arrow held in place next to the bow by one hand. In this example the bent bow corresponds to the hole in the ground. The arrow is quite stable when held against the bow with only one hand and you do not have to hold the feathered and notched end of the arrow against the bent string with your other hand. The arrow does not jump or bend away but "sticks" to the bow string. A bow and arrow could be considered to be the earliest form of tensegrity with the arrow as the "supported" stick and the bow as the base on which the arrow is supported by the bow string. Whether you consider this entire structure a one or a two stick tensegrity is a matter of viewpoint. A bow by itself with its string stretched taunt but with no arrow in place could be considered to be a one stick tensegrity. A mathematician would probably consider it a "trivial" or "degenerate" case.

A well written and illustrated exposition of this principle employing two sticks and three guy wires can be found on page 6 of "Geodesic Math and How to Use It" by Hugh Kenner, 2nd paperback edition, 2003, University of California Press [2]. The book is available through Amazon.com. If you Google the title and author you will also find the book available on-line at http://books.google.com where you can view page 6 and a little beyond. Unfortunately, Kenner does not carry through with this principle in his analysis of tensegrity prisms which follows page 6, preferring instead to employ calculus to establish something he aptly names "The Constant Twist Angle Theorem." While calculus is doubtless much more persuasive to academically trained mathematicians and engineers, it misses making clear certain fundamental geometrical facts that an approach based on synthetic projective and incidence geometry reveals. Ideally, of course, the two approaches, of calculus and of descriptive incidence geometry leading to plane geometry, should be combined. I should note that Kenner bases this analysis directly on my 1967 Pratt Master of Industrial Design thesis [6] which he credits in a footnote.

The plan views of tensegrity prisms can be reduced to the elementary Euclidean geometry of ruler and compass. In fact, the very first figure in the Dover edition of Heath's Elements of Euclid [is the beginning of the plan view of a three stick tensegrity prism. And, as previously stated, it turns out that the plan views of all tensegrity prisms are invariant no matter how the overall ratio of height to width varies. That is, if the stick lengths are all the same, if the top and bottom string lengths are all the same, and if the side string lengths are all the same, then the plan view never varies and consists of a simple arrangement of circles of equal radius. The same sort of pattern of equal radius circles applies to all tensegrity prism with the only difference being the number of circles. The "constant twist angle theorem" can be derived from these plan views. It further turns out that the angles of all tensegrity prisms in plan view are rational fractional parts of a circle if you take the circumference of a circle to be equal to ***one full turn*** instead of (2 pi r) as is normally done using radian measure. Then it becomes apparent that the angular measurement of these circumscribing circles is naturally quantized being always some integral multiple of a basic "unit" angle. In order to establish these results it is necessary to show that most, if not all, tensegrities of whatever overall shape, certainly the ones based on right prisms, can be analyzed in terms of intersecting "planes of force". Then everything falls into place.

2. The Spoked Wheel Triangle Replacement

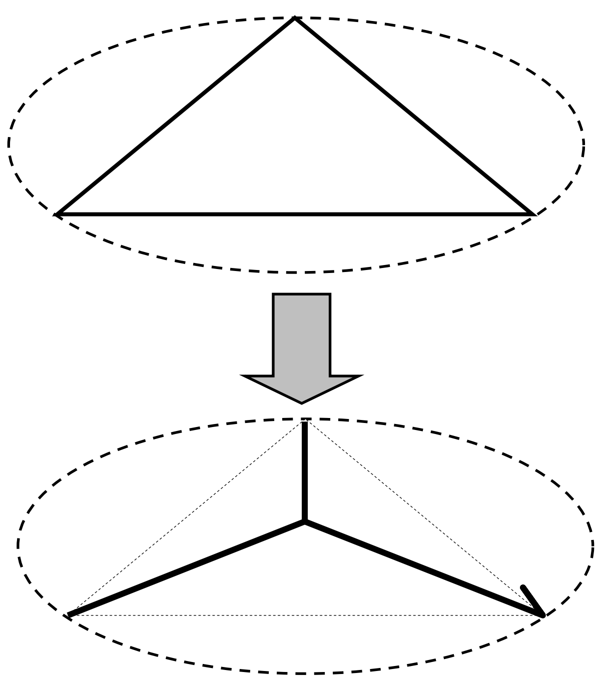


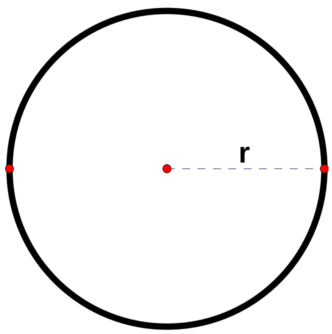
Fig. 4. A triangle of top or bottom strings replaced with a spoked wheel of top or bottom strings.

If the sticks composing tensegrity prisms are to have their end points attached and held in place by only two tensional elements or strings, then the top and bottom strings must be connected and gathered together at their mid points like the hub of a spoked wheel. In a three stick tensegrity, for instance, it is no longer possible to exclusively connect all top and bottom cables from one stick end to the next around the perimeter of a triangle; rather the three mid points of the top and bottom cables must meet at a single common hub like the spokes of a bicycle wheel. (See Fig. 4.) Instead of having a top and bottom triangle with three edges, you have a top a bottom triangle composed of three spokes. The triangle vertices remain the same. The total number of top and bottom strings remains the same, but they are differently arranged. This pulling together of midpoints of top and bottom strings can be performed on tensegrity prisms composed of 4 or more sticks creating spoked wheels of 4 or more spokes. I claim that when this conversion is made, the overall form of the particular tensegrity prism is not changed. When this restringing is made a tensegrity prism can be seen to be composed of intersecting "planes of force" where every stick lies at the intersection of two planes and each group of "top" and "bottom" interconnected end points lie within common end planes.

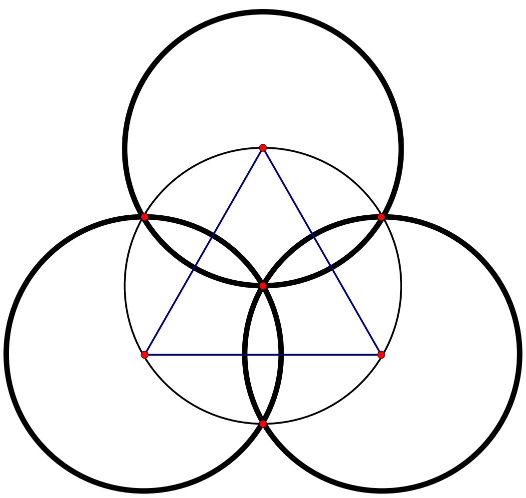
Thus in a three stick tensegrity prism there are two top and bottom planes of force and 6 internal planes of force, two for each of the three sticks, yielding a total of 8 planes in all in a three stick tensegrity prism. I claim this "spoked wheel" re-stringing clearly reveals the "planes of force" within a tensegrity without changing its overall form thus making it is possible to use the geometry of intersecting planes to analyze and establish the dimensions and angular relationships of any n stick tensegrity prism, without having to know anything about the forces within the individual tension or compression elements. You just have to pull your strings or wires tight until the structure is in an equilibrium position.

However, it is not clear whether the types of towers and compound tensegrity structures designed and built by Kenneth Snelson, can all be re-strung with only two strings per stick end and maintain their overall forms. I tend to think that some of them cannot. Clearly, the multiple stages of structure like Snelson's Needle Tower where one three stick prism stands on the "top string" cables of the three stick prism below it requires three tensional elements per stick end as long as the sticks are not allowed to touch and pivot on each other end to end, but are required to float and be supported solely by cables. But, I conjecture that all individual or "atomic" tensegrities can be strung and analyzed in the way I have described. The word here that "begs the question" is "individual". What exactly is meant by an individual tensegrity? It is somewhat like a prime number which cannot be decomposed into any elements but 1 and itself whereas a composite number can be decomposed into prime factors. I propose that a minimum individual or "prime" tensegrity is a minimum tensionally integral structure which cannot have any of its tension or compression elements cut without overall collapse. For instance if you cut one stick or one string in a three stick tensegrity prism, then the whole thing collapses. Every tension and compression element is essential. I believe that the sort of re-stringing where circumferential convex polygons of string are replaced with equivalent spoke structures can be used to predict and analyze the forms of compound multi-strung structures. But, as I say, this is a conjecture. I have not actually performed this experiment. It has not been physically demonstrated nor mathematically proven. This kind of spoked stringing loses a great deal of mechanical advantage and is less stable physically than three or more strings per stick end. Clearly, three or more strings per stick end provide much more stability and much greater mechanical advantage. You have better triangulation with less acute angles, but for purposes of analysis, two strings per stick end reveals certain underlying geometric facts

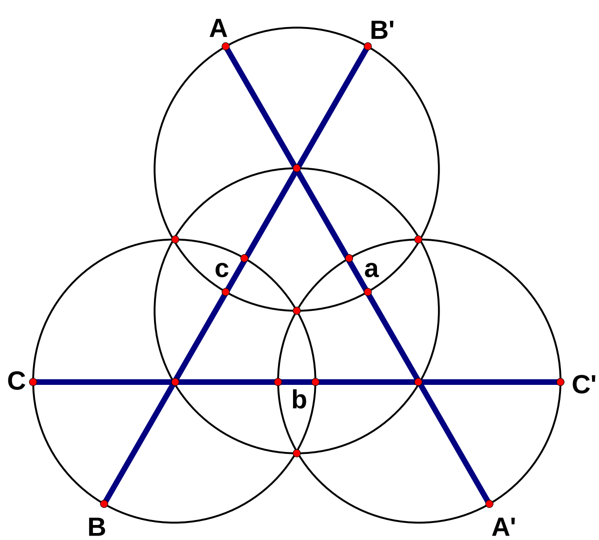
3. The Circle Drawing Algorithm



Step 1. Start with a circle of radius *r*.

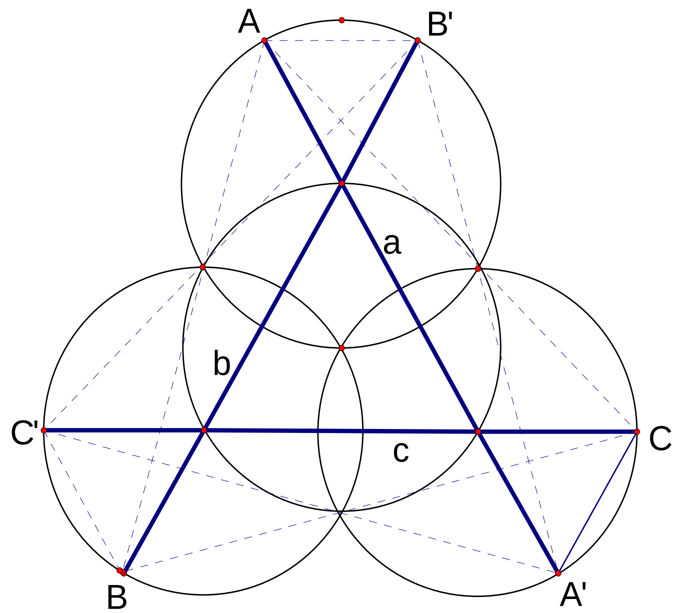


Step 2. Draw three more circles of radius *r* with centers spaced by thirds around the circumference of the original circle.

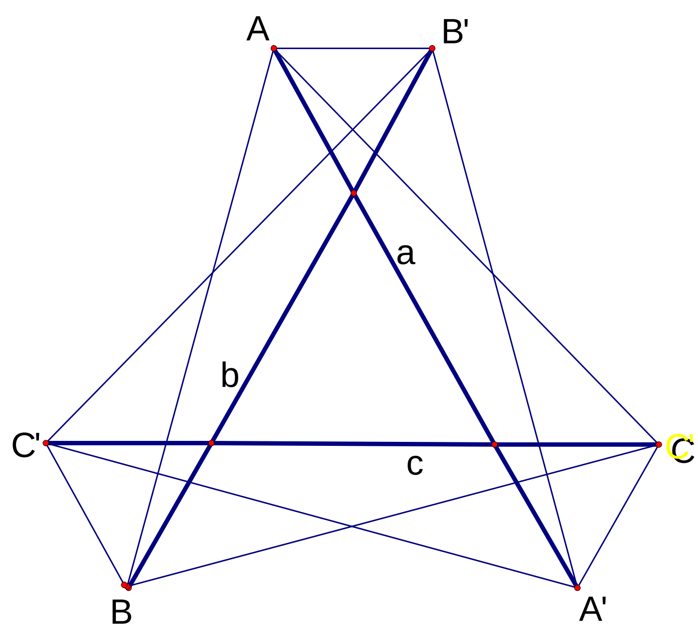


Step 3a. Extend the three sides of the central triangle until they meet the circumferences of the outside circles.

Step 3b. …and then label the line and circle interseciontion points as shown above as A & A′ for line a, B & B′ for line b, and C & C′ for line c. Lines a, b, and c represent the three sticks of any three stick tensegrity as seen from above (or below) in plan view. Depending on the spiral overlap of the sticks, the overall structure can be either left or right handed.

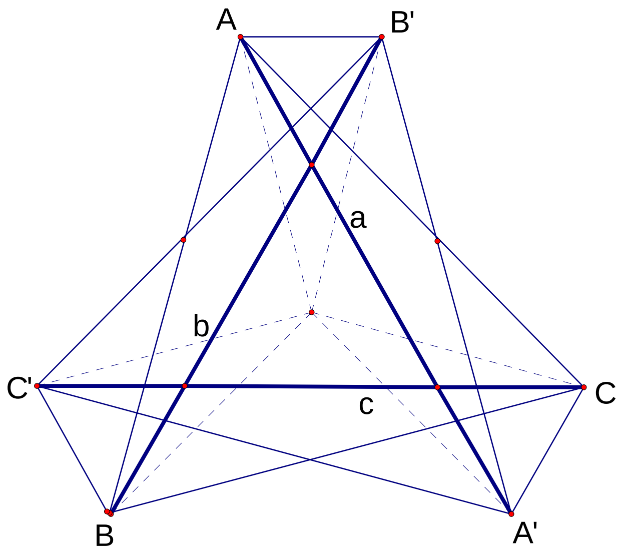


Step 4. Draw triangles ABC and A′B′C′. These represent the top and/or bottom triangles of strings. Draw line segments AB′, BC′, and CA′. These represent the side strings.

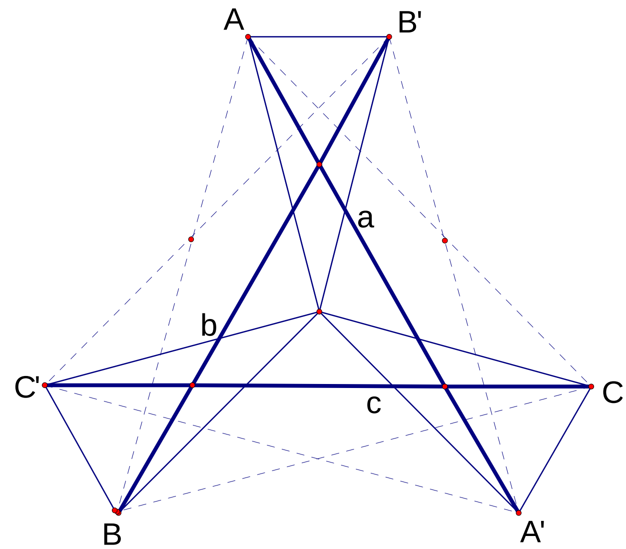


Step 5. Erase the guide circles. What remains is the plan view of a three stick tensegrity, either right of left handed depending how you choose to spiral and overlap the three sticks.

The two plan views (Steps 6 and 7) are the same overall shape whether the tops and bottoms are supported by triangles of string of by central spoked wheels of string. The end points A & A′, B & B′, and C & C′ of the three sticks do not shift positions in relationship to each other. Thus the angular and linear measurements remain the same between the 2 drawings.



Step 6a. Replace the top and bottom triangles of strings with a pair of top and bottom three spoked wheels of strings whose centers lie at the center of the plan view of the overall structure. Notice the change from thin solid lines to thin dashed lines and vice versa between these two drawings representing this transition.



Step 6b. Here the dashed lines represent the replaced triangles.

This circle drawing algorithm can be generalized and applied to a tensegrity prism of any number of sticks, say 4, 5, or 6 (see below) or 92. The plan view drawing for 92 sticks is quite impressive. Corresponding physical structures for these numbers of sticks do exist and can be built.

4. Three and Four Stick Plan Views Side by Side

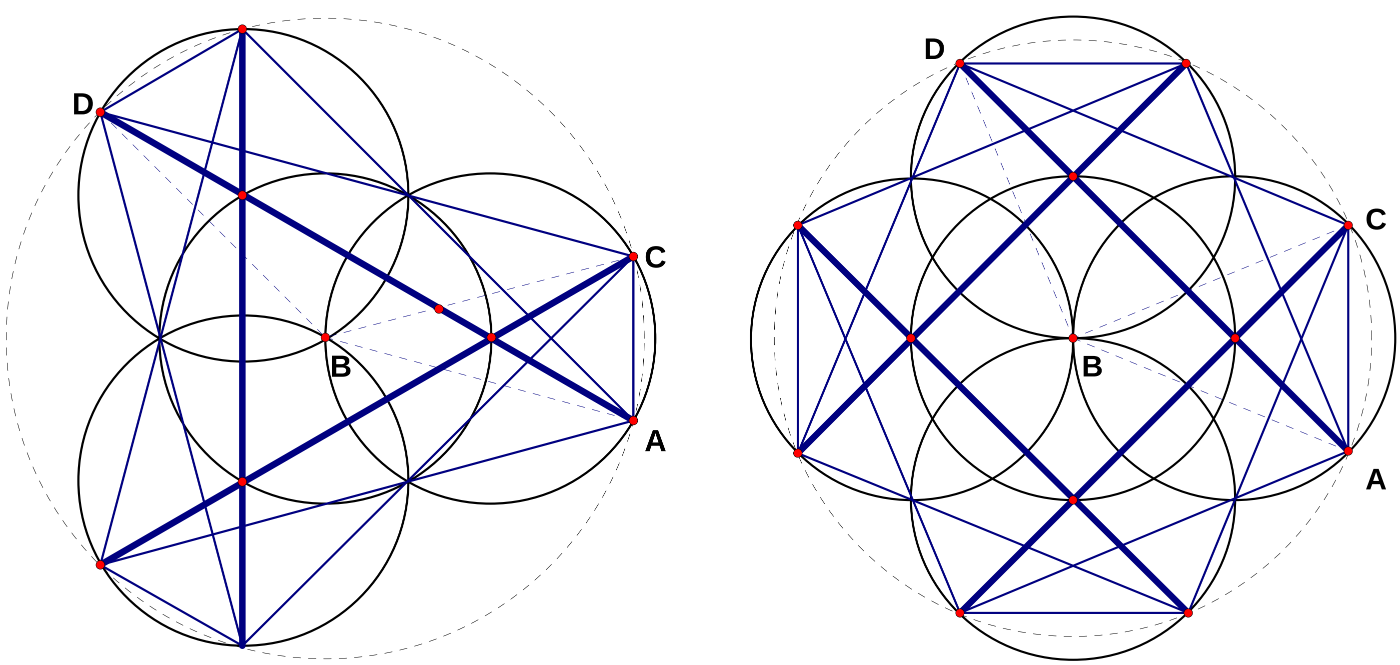


Fig. 5. Three and Four Stick Plan Views Side by Side.

Least angle = side string angle ABC = 30°
Top or bottom string angle CBD = 120°
Stick angle ABD = 150°

Least angle = side string angle ABC = 45°
Top or bottom string angle CBD = 90°
Stick angle ABD = 135°

5. Five and Six Stick Plan Views Side by Side

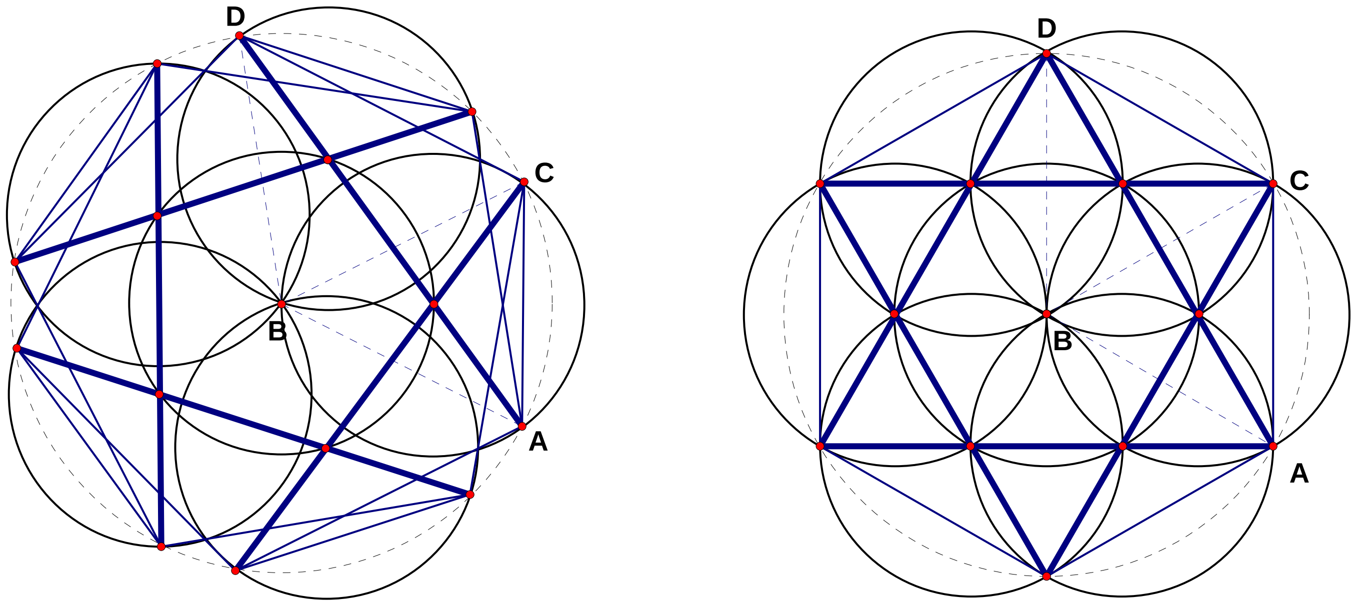


Fig. 6. Five and Six Stick Plan Views Side by Side.

Least angle = side string angle ABC = 54°
Top or bottom string angle CBD = 72°
Stick angle ABD = 126°

Least angle = side string angle ABC = 60°
Top or bottom string angle CBD = 60°
Stick angle ABD = 120°

From these simple formulas it follows that within any tensegrity prism the stick angle plus the side string angle always adds up to one half of a circle or 180°. For example, writing (2πr) for one complete revolution of a circle of 360° and using * for the multiplication sign, then the following formulas may be summarized for 3, 4, 5 and 6 stick structures as displayed above. This pattern can be extended for any number of sticks and naturally falls into a pattern with a period of 4.Thus first row (3, 4, 5, 6) sticks; second row (7, 8, 9, 10) sticks; third row (11, 12, 13, 14) sticks; fourth row (15, 16, 17, 18) sticks and so on. Arranged in a four columns this period four pattern bears some resemblance to the periodic table of the elements which has a count of 8 across in its middle group of elements, and curiously the members of each column have a common geometric form. They belong to the same "family". For instance in the 6, 10, 14, and 18 stick forms that end the first four rows, the upper end of one stick is directly perpendicularly above the lower end of another stick. You could say these forms have closed in on themselves.

If $n = 3$, then

$$\text{side string angle} = \frac{3-2}{4 \cdot 3} = \frac{1}{12} \text{ of circle} = 30° \tag{1}$$

and
$$\text{stick angle} = \frac{3+2}{4 \cdot 3} = \frac{5}{12} \text{ of circle} = 150° \tag{2}$$

hence
$$\frac{1}{12}(2\pi r) + \frac{5}{12}(2\pi r) = \frac{1}{2}(2\pi r) = 180° \tag{3}$$

If $n = 4$, then

$$\text{side string angle} = \frac{4-2}{4 \cdot 4} = \frac{2}{16} \text{ of circle} = 45° \tag{4}$$

and
$$\text{stick angle} = \frac{4+2}{4 \cdot 4} = \frac{6}{16} \text{ of circle} = 135° \tag{5}$$

hence
$$\frac{2}{16}(2\pi r) + \frac{6}{16}(2\pi r) = \frac{1}{2}(2\pi r) = 180° \tag{6}$$

If $n = 5$, then

$$\text{side string angle} = \frac{5-2}{4 \cdot 5} = \frac{3}{20} \text{ of circle} = 54° \tag{7}$$

and
$$\text{stick angle} = \frac{5+2}{4 \cdot 5} = \frac{7}{20} \text{ of circle} = 126° \tag{8}$$

hence
$$\frac{3}{20}(2\pi r) + \frac{7}{20}(2\pi r) = \frac{1}{2}(2\pi r) = 180° \tag{9}$$

If $n = 6$, then

$$\text{side string angle} = \frac{6-2}{4 \cdot 6} = \frac{4}{24} \text{ of circle} = 60° \tag{10}$$

and
$$\text{stick angle} = \frac{6+2}{4 \cdot 6} = \frac{8}{24} \text{ of circle} = 120° \tag{11}$$

hence
$$\frac{4}{24}(2\pi r) + \frac{8}{24}(2\pi r) = \frac{1}{2}(2\pi r) = 180° \tag{12}$$

From inspecting these formulas some interesting numerical patterns or progressions are apparent. For instance, considering the series of side string angles starting with the first side string angle for n = 3 sticks, we have 1/12, 2/16, 3/20, 4/24 where each numerator increases by 1 and each denominator increases by 4. Similarly for the stick angles we have 5/12, 6/16, 7/20, 8/24. Again the +1 and +4 progression. If we add the corresponding terms in each of these sequences we have 6/12, 8/16, 10/10, 12/24 which, of course, all reduce to 1/2, that is, 1/2 of a complete circle or 180°

It is worth noting that all the angular values in these structures are rational. They are whole number fractions controlled by the single integer value n. One could argue that this system of structures exhibits "natural quantization" akin to the standing waves of electrons sometimes supposed to surround atoms. The only requirement to guarantee this natural quantization is that in any particular structure all the sticks are the same length, all the top and bottom strings are the same length, and all the side strings are the same length, whatever these three different lengths may be.

6. Axioms of Incidence

Axioms of incidence are constantly used implicitly in reasoning about tensegrities, but we do not make a great deal of explicit reference to them in this rather informal presentation. Some of these axioms are as follows.

Two points determine a line. That is, through any two points a line may be drawn or, better yet, stretched tight.

Two lines determine a point. That is, the intersection of two non-parallel lines that lie in the same plane is a point.

Two intersecting lines placed anywhere in a 3-space determine a plane.

Similarly, two parallel lines placed anywhere in a 3-space determine a plane. In projective geometry two parallel lines are considered to meet at a point at infinity and this is considered to be a special case of two intersecting lines.

Three points determine a plane. That is, three non-collinear points placed anywhere in a 3-space determine a plane.

Three planes determine a point. That is, three non-parallel planes placed anywhere in a 3-space intersect in one and only one point.

The above axioms are arranged more or less as dual pairs where the words point and line or point and plane may be interchanged, but we will not get into a detailed discussion of dualism in projective geometry here. Suffice it to say that in the subject of tensegrities some other kinds of dualisms do occur between sticks and strings, push and pull, and inequalities of lengths which are greater or less than each other, etc.

We should add, for the sake of completeness, that a line and a plane that are not parallel and not coincident with each other intersect in one, and only, one point.

7. Axioms of Anchoring:

One area where "tensile geometry", to coin a phrase, differs from standard geometry is the requirement for something I call axioms of anchoring which have to do with the basic notions of push and pull. Tensegrities may be considered to be the study of applied push and pull. The axioms we give below all have to do with pull, but there are some equally interesting, if less often used, axioms of push. The very first axiom which is a "push" axiom and may really be **Axiom Zero**, is, "You can't push a string. You can only pull it!"

It takes two opposing collinear line segments in tension to anchor or hold a point in a one dimensional space. The two line segments are joined together at the anchored point and lie in one line.

Fig. 7. Two lines pulling on a point between them.

It takes three opposing non-collinear line segments in tension to anchor or hold a point in a two dimensional space, that is to say, a plane. The three line segments are joined together at the anchored point and lie in a common plane. This configuration lies within a triangle within the plane, with the vertices of the triangle as the "outside" end points of the line segments.

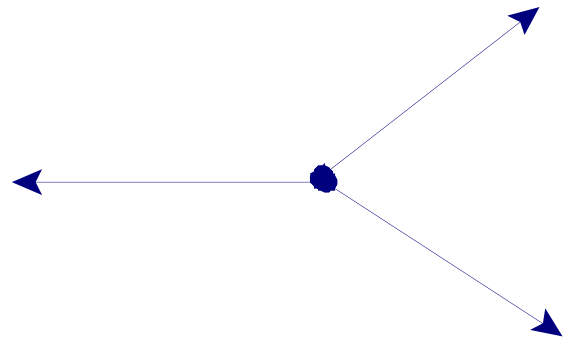


Fig. 8. Three lines pulling on a point, the "spoked wheel".

It takes four opposing non-collinear line segments in tension to anchor or hold a point in a three dimensional space. The four line segments are joined together at the anchored point and lie in one 3-space. The bounding volume is a tetrahedron with its vertices at the four "outside" end points of the strings or lines.

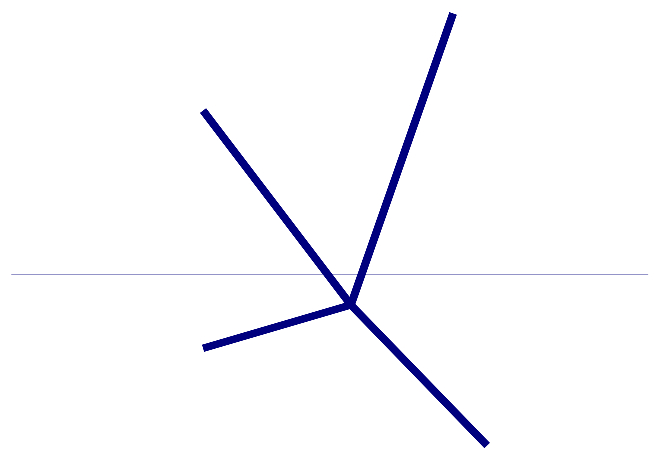


Fig. 9. Four lines pulling on a common end point.

And by analogy one may suppose that it takes five opposing non-collinear line segments in tension to anchor or hold a point in a four dimensional space. The five line segments are joined together at the anchored point and lie in one 4-space volume. I'm not sure what the shape of this containing volume would be. It is clearly not a hypercube. That has too many vertices.

Thus the number of opposing, joined-together, non-collinear line segments required to anchor or hold a point in n-dimensions is $(n + 1)$. By opposing it is meant that the line segments are pulling away from and against each other and that their "outer" or "free" end points form a convex set surrounding the joining point that they are anchoring. Presumably the outside or free points must in turn be tied or anchored to some external and surrounding framework in order for such an arrangement to work.

The above axioms lead naturally to what is perhaps the single most important tensegrity axiom of all, namely, the closed loop axiom.

8. The Closed Loop Axiom

Without exception all the strings within a tensegrity must lie in a continuous network of closed loops which surrounds and lies essentially outside of the sticks. Thus the sticks must be surrounded by and enclosed in an envelope of tension.

Thus the end points of the sticks are joined to each other in a cyclic or looping structure in such a way that the entire structure becomes self supporting and integral. Speaking loosely, the structure surrounds itself. The stick end points don't have to be anchored to some external global foundation structure or outside frame of reference. A tensegrity consists of one or more **closed** loops of tension bearing down on one or more islands of compression. The emphasis must be equally on **closed** *and* **loop**. The one apparent exception to this complete single closed loop requirement is a bow, as in a bow and arrow, where there is a single string bending a single compression element, but even here, the string and the bow do form a closed loop of forces mostly in tension. It is the *feedback* of forces around these closed loops that make a tensegrity fundamentally different from a simple post and beam compression structure like the Parthenon, say, and give it its "integrity". If you cut one or more of these loops the entire structure collapses.

9. Areas for Further Inquiry

As mentioned above the work of Rankine and Maxwell on the calculation of stresses in frameworks appears to anticipate this sort of geometry. It is "an extension of the well-known triangle of forces in statics" to quote one author. See page 169 of "James Clerk Maxwell: Physicist and Natural Philosopher" by C.W.F Everitt [5]. Everitt mentions, for instance, Maxwell's geometrical discussion entitled "On Reciprocal Figures and Diagrams of Forces" and the work of the Italian projective geometer Luigi Cremona on the graphical analysis of forces.

I was led to the work of Cremona as well as Maxwell by an article entitled "Structural Topology, or The Fine Art of Rediscovery" by Henry Crapo in The Mathematical Intelligencer , vol. 19, no.4, Fall 1997. Cremona's book which is referenced in this article is "Graphical Statics: Two Treatises on the Graphical Calculus and Reciprocal Figures in Graphical Statics" Oxford Univ. Press, 1890. It is (or was) available in reprint from The University of Michigan Printing Services Copy Center. I got a hard copy from them for the ridiculously low price of $17.73 plus $6.00 shipping and handling.

Observation: Maxwell's stability criterion: (3n - 6) lines are needed for stability of a figure of n points. (See page 170 of the Everitt book [5] for this formula.) For a three stick structure with 6 points, that's 2 end points per stick. Thus we have (3*6 - 6) = (18 - 6) = 12 lines. And there are a total of 12 lines in a three stick structure. They are the 3 sticks, 3 side strings, 3 top strings, and 3 bottom strings for a total of 12 lines where a line can be either a stick or a string. What about the extra two points formed by a spoked wheel of lines if we replace top and bottom triangles with top and bottom three spoked stars? Do we count those two extra points for a total of eight points? I'm not sure. I think we have two kinds of points, string to string joining or tie points and string to stick joining or tie points.

10. Conclusion

"Art Gallery Theorems and Algorithms" by Joseph O'Rourke, Oxford Univ. Press, 1987. On pages 253-254 of this book O'Rourke discusses untetrahedralizable polyhedra. On page 254 is shown "Fig. 10. 1. Schonhardt's untetrhedralizable polyhedron, constructed by twisting the top of a triangular prism (a) by 30 degrees, producing (b), shown in top view [c]; a twist of 60 degrees would cause face intersections (d)." Top view [c] is precisely the plan view of a **three stick tensegrity**. It even has the correct twist angle, 30 degrees. I don't think anyone realized that this "is a tensegrity". It is interesting to note that if you build a model of the planes of a three stick tensegrity, there is a virtual octahedron at the center. O'Rourke writes that Schonhardt proved his result in 1928.

On page 56 of H. S. M. Coxeter's "Regular Complex Polytopes" Cambridge Univ. Press, 1974, and again in the revised 2nd edition, 1991, there is a diagram labeled "The diagram for (5*14)" which appears to be the same as a three stick plan view. It is interesting to note that this is in a chapter on Frieze Patterns and the diagram has something to do with periodicity in which the numbers 3 and 6 figure prominently.

On page 71 of "A Treatise on Plane & Advanced Trigonometry" by E. W. Hobson, Dover Publications reprint of the 1928 edition (the first edition was 1891) there is a diagram that is strongly suggestive of the plan view of a three stick tensegrity. This diagram appears in a section on The Circular Functions of Submultiple Angles. It is abundantly clear that there are lots of submultiple angles in the plan views of twisted tensegrity prisms.

References:

[1] "Geodesic Math and How to Use It" by Hugh Kenner, University of California Press, 2nd paperback edition, reprinted 2003. Currently available.

[2] "An Introduction to Tensegrity" by Anthony Pugh, University of California Press, 1976.

[3] "The Geometry of Incidence" by Harold L. Dorwart, Prentice Hall, 1966. A very good introduction for the layman to the basic concepts of projective geometry. This is the book that made me realize that tensile structures (as I called them at the time) were specializations of projective geometric forms.

[4] "The Thirteen Books of Euclid's Elements" Translated by Sir Thomas L. Heath. 2nd. Revised edition. Dover Publications, 1956. Still in the Dover math catalog last time I looked.

[5] "James Clerk Maxwell: Physicist and Natural Philosopher" by C.W.F Everitt, Charles Scribner's Sons, 1975.

[6] "A Report on an Inquiry into the Existence, Formation and Representation of Tensile Structures" by Roger S. Tobie (Master's thesis in Department of Industrial Design, Pratt Institute: 1967). This paper is basically a condensation of that thesis.