"Fifth" Force
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Is a "fifth" force really necessary? Obviously, a new force is much more exciting than finding an explanation for the measured effect within another force description. A new force may be more tenable because it may appear not to compete with existing forces. A correction to existing forces may lack excitement and must certainly be shown to be compatible with existing forces where they are measured to great accuracy. A correction to an existing force is usually difficult to find and may go against the preference of many. But to assume there is an additional force is to assume its independence and would then necessitate yet another force to be "unified".

Accurate measurements show that the gravitational force of the Earth differs from Newton's Law at close range. More specifically, the difference in the Earth's gravitational field over a difference of height in a deep well is not the same as predicted by Newton's Law. This leads to a simple choice; either Newton's Law of gravity needs to have a correction or an independent fifth force is needed to explain the difference. In the past when we found that the proton-proton scattering data differed from the Coulombic predictions we opted for an independent force and have had the fun of searching for a method of unifying electromagnetism and the strong force every since. We could make that same choice here, or we could investigate the difference in the prediction of the non-singular potential and Newton's law of gravity. To do this we need to look at the gravitational attraction on a mass in a well deep down from the surface of the Earth.

Freshman texts on physics typically show how to calculate the gravitational influence of a thin spherical shell on a mass both inside and outside of the shell. This is the procedure we need here because a mass in a deep well feels the influence of both the mass of the Earth interior to it and in the shell of the Earth exterior to it where the shell thickness is the depth of the well. If we recall the procedure for this using the Newtonian potential, then we remember that, for the 1/r potential, all of the mass interior to the test mass attracts it as if the mass were located at the center of the Earth. On the other hand, we recall that there is no gravitational influence due to the mass in the outer shell which is exterior to the test mass in the deep well. For a potential which differs from the 1/r Newtonian potential these conclusions may not be true. Indeed, one’s first suspicion is that they are not the correct conclusions. What we now need to do is to calculate the influence on a test mass both outside and inside of gravitating mass.

First, suppose we calculate the gravitational influence on a mass, m, exterior to a thin, uniformly dense, spherical shell (see Figure 1). We assume that each particle of the shell, dM, exerts a force on the test mass which is given by the non-singular force, or

$$d\bar{F} = -\bar{r}\frac{m(dM)G}{r^3}\left(1-\frac{\lambda}{r}\right)e^{\frac{-\lambda}{r}}$$

The small part of the shell at A attracts m with a force $\boldsymbol{F_1}$. A small part of equal mass at B, equally far from m but diametrically opposite A, attracts m with a force $\boldsymbol{F_2}$. The resultant of these forces on m is $\boldsymbol{F_1} + \boldsymbol{F_2}$. Notice, however, that the vertical components of these two forces cancel one another and that the horizontal components, $F_1 cos\alpha$ and $F_2 cos\alpha$, are equal. By dividing the spherical shell into pairs of particles like these, we can see at once that all transverse forces having an upward component on m cancel in pairs. To find the resultant force on m arising from the shell, we need consider only horizontal components.

Let us take a circular strip of mass as the element of mass of the shell labeled dS in the figure. It's length is $2\pi(rsin\theta)$, its width is $rd\theta$, and its thickness is t. hence, it has a volume

$$dV = 2\pi tr^2 sin\theta d\theta.$$

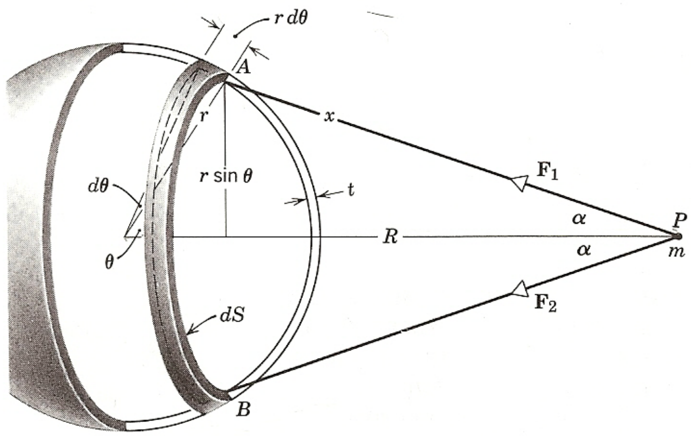

Figure 1. Gravitational attraction of a section, dS, of a spherical shell of matter on m.

Let us call the density ρ, so that the mass within the strip is

$$dM = \rho dV = 2\pi t\rho r^2 sin\theta d\theta.$$

The force exerted by dM on the particle of mass m at P is horizontal and has the value

$$dF = G\frac{mdM}{x^2}\left(1-\frac{\lambda}{x}\right)e^{\frac{-\lambda}{x}}cos\alpha = 2\pi t\rho r^2 mG\frac{sin\theta d\theta}{x^2}\left(1-\frac{\lambda}{x}\right)e^{\frac{-\lambda}{x}}cos\alpha.$$

The variables x, α, and θ are related. From the figure we see that

$$cos\alpha = \frac{R - rcos\theta}{x}.$$

Since by the law of cosines,

$$x^2 = R^2 + r^2 - 2Rrcos\theta,$$

we have

$$rcos\theta = \frac{R^2 + r^2 - x^2}{2R}.$$

Differentiating the previous equations we obtain

$$2xdx = 2Rrsin\theta d\theta$$

or

$$sin\theta d\theta = \frac{x}{Rr}dx.$$

Now we may use these results to obtain the element of force as

$$dF = \frac{\pi t\rho rmG}{R^2}\left(\frac{R^2 - r^2}{x^2} + 1\right)\left(1-\frac{\lambda}{x}\right)e^{\frac{-\lambda}{x}}dx.$$

This is the force exerted by the circular strip dS on the particle m. We now need to integrate this differential element of force as x ranges from R-r to R+r. we must watch the lower limit as the function may become infinite at this limit. This makes us suspect that this integration will involve the Exponential Integral function. The integral we need to solve is then

$$F = \int_{R-r}^{R+r} dF = \frac{\pi t\rho rmG}{R^2}\int_{R-r}^{R+r}\left(\frac{R^2 - r^2}{x^2} + 1\right)\left(1-\frac{\lambda}{x}\right)e^{\frac{-\lambda}{x}}dx.$$

A potential problem with this integral is that the lower limit of R-r appears to make the integrand go to infinity and rule out a solution. However, the nature of the exponential brings the Exponential Integral into the picture and shows that the force remains finite. The stepwise integration of the differential force is done in the Appendix and results in the solution

$$\begin{aligned} F = -\frac{mMG}{R^2}\left[\!\!\left[\left[e^{\left(\frac{-\lambda}{R+r}\right)} + e^{\left(\frac{-\lambda}{R-r}\right)}\right] - \left(\frac{R^2 - r^2}{\lambda}\right)\left[e^{\left(\frac{-\lambda}{R+r}\right)} - e^{\left(\frac{-\lambda}{R-r}\right)}\right]\right.\right. \\ \left.\left. - \lambda\left\{ln\left[\frac{R+r}{R-r}\right] + \sum_{N=1}^{N=\infty}\frac{(-\lambda)^N}{N \cdot N!}\left[\frac{1}{(R+r)^N} - \frac{1}{(R-r)^N}\right]\right\}\right]\!\!\right]. \end{aligned}$$

Though this looks like an imposing result, and it makes calculations difficult, one may easily show that this reduces to the classical Newtonian result when λ vanishes.

In order to determine the gravity on a test mass that is below the Earth's surface we also need to calculate the possible contribution due to the hollow spherical shell of mass existing between the test mass and the Earth's surface which has a thickness equal to the depth of the test mass.

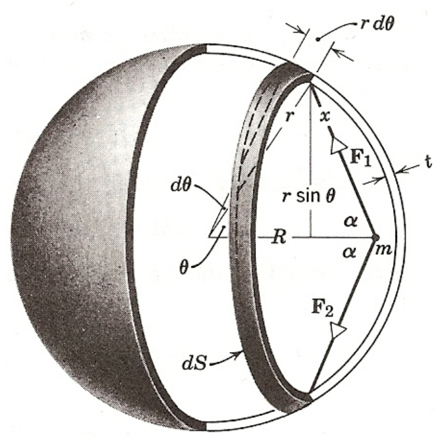

Figure 2. Gravitational attraction of a section, dS, of a spherical shell of matter on m.

In Figure 2, where the test mass is inside the spherical shell, we may see that now *R* is smaller than *r*. the limits of our integration over x is now *r-R* to R+r. so that our integration may be stated as

$$F == \frac{\pi t \rho r m G}{R^2}\int_{r-R}^{R+r}\left(\frac{R^2 - r^2}{x^2} + 1\right)\left(1 - \frac{\lambda}{x}\right)e^{\frac{-\lambda}{x}}dx.$$

This change in the limits of integration results in

$$\begin{aligned} F_{inside} = \frac{mMG}{R^2}\left[\!\!\left[\left(\frac{r^2 - R^2}{\lambda}\right)\left[e^{\left(\frac{\lambda}{R-r}\right)} - e^{\left(\frac{-\lambda}{R+r}\right)}\right]\right.\right. \\ \left.\left. + \lambda\left\{ln\left[\frac{R+r}{r-R}\right] + \sum_{N=1}^{N=\infty}\frac{(-\lambda)^N}{N \cdot N!}\left[\frac{1}{(R+r)^N} - \frac{1}{(r-R)^N}\right]\right\}\right]\!\!\right]. \end{aligned}$$

This may be seen to correspond to the classical Newtonian result where the force on a test mass inside of a spherical shell of mass is zero when λ goes to zero.

Here the force on a test mass inside a shell experiences a small force outward toward the shell.

Conclusion

The combined result of the non-singular force on a test mass outside of a sphere being a little less than Newtonian and the force on the test mass inside a hollow sphere being positive drawing the test mass outward toward the shell means that a measurement of the gravity on a test mass below the Earth's surface should be less than the Newtonian prediction. On the other hand, measurements made above the Earth's surface should correspond to the Newtonian prediction. This, of course is just what all of the measurements reported under the research into a Fifth Force showed. Thus, the non-singular gravitational potential explains these data without resorting to a new, independent force.

Appendix

This Appendix contains the steps to determine the solution to the integral

$$F = \int_{R-r}^{R+r} dF = \frac{\pi t \rho r m G}{R^2} \int_{R-r}^{R+r} \left(\frac{R^2 - r^2}{x^2} + 1 \right) \left(1 - \frac{\lambda}{x} \right) e^{\frac{-\lambda}{x}} dx.$$

This may be done by considering the integral

$$f = \int_{R-r}^{R+r} \left(\frac{R^2 - r^2}{x^2} + 1 \right) \left(1 - \frac{\lambda}{x} \right) e^{\frac{-\lambda}{x}} dx.$$

Make the substitution $u = \lambda / x$ so that our integral may be written as

$$f = \int_{x=R-r}^{x=R+r} \left(\frac{R^2 - r^2}{\lambda^2} u^2 + 1 \right) (1 - u) e^{-u} \left(-\frac{\lambda}{u^2} du \right).$$

This may be rewritten as

$$f = \int_{x=R-r}^{x=R+r} - \left[\left(\frac{R^2 - r^2}{\lambda} - \frac{R^2 - r^2}{\lambda} u - \frac{\lambda}{u} + \frac{\lambda}{u^2} \right) e^{-u} \right] du.$$

This may now be integrated term by term then summed and x substituted for u to get the final result of

$$F = -\frac{mMG}{2R^2} \left[e^{\left(\frac{-\lambda}{R+r} \right)} + e^{\left(\frac{-\lambda}{R-r} \right)} \right] - 2 \left(\frac{R^2 - r^2}{\lambda} \right) \left[e^{\left(\frac{-\lambda}{R+r} \right)} - e^{\left(\frac{-\lambda}{R-r} \right)} \right] - 2\lambda \left\{ ln \left| \frac{R+r}{R-r} \right| + \sum_{N=1}^{N=\infty} \frac{(-\lambda)^N}{N \cdot N!} \left[\frac{1}{(R+r)^N} - \frac{1}{(R-r)^N} \right] \right\}.$$