On "hidden momentum" of magnetic dipoles

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This paper examines the problem of "hidden" momentum for quasi-static macroscopic systems. We have analysed a number of particular physical problems and shown that introducing of "hidden" momentum is strongly required to fulfil the energy-momentum conservation law.
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1. Introduction

It is known that the electromagnetic (EM) field has a momentum density $\vec{p}_{EM} = \varepsilon_0 \left(\vec{E} \times \vec{B} \right)$ (in SI units), which is customarily applied to both free and bound (non-radiating) EM fields [1]. Then the total momentum of a bound field is computed by integration $\vec{p}_{EM}$ over all free space V:

$$\vec{P}_{EM} = \varepsilon_0 \int_V \left(\vec{E} \times \vec{B} \right) dV \, . \tag{1}$$

The law of conservation of total momentum (electromagnetic $\vec{P}_{EM}$ plus mechanical $\vec{P}_M$) implies that it should be constant for any isolated system of free charged particles: $\vec{P}_M + \vec{P}_{EM} = const$, and

$$\frac{d\vec{P}_M}{dt} = -\frac{d\vec{P}_{EM}}{dt} = -\varepsilon_0 \frac{d}{dt}\int_V \left(\vec{E}\times\vec{B}\right) dV . \quad (2)$$

Eq. (2) describes the self-force, acting on this isolated system due to a violation of Newton's third law in EM interaction. If we employ only the interaction part of EM momentum in *rhs* of Eq. (2), we can rewrite it in a more convenient form (see, *e.g.* [2])

$$\varepsilon_0 \int_V \left(\vec{E}\times\vec{B}\right) dV = \sum_{i=1}^{N} q_i \vec{A}_i , \quad (3)$$

where N>1 in the number of charged particles within the considered isolated systems, q_i is the charge of the i^{th} particle, and $\vec{A}_i$ is the vector potential, created by (N-1) particles at the location of particle i. Then Eq. (2) can be rewritten in the form

$$\frac{d}{dt}\sum_{i=1}^{N} \vec{P}_{Mi} = -\frac{d}{dt}\sum_{i=1}^{N} q_i \vec{A}_i , \quad (4)$$

where we presented the total mechanical momentum $\vec{P}_M$ as the sum of mechanical momenta for each particle. Eq. (4) shows that the canonical momentum of the system

$$\vec{P}_G = \sum_{i=1}^{N}\left(\vec{P}_{Mi} + q_i \vec{A}_i\right) \quad (5)$$

is conserved for an isolated system. Eq. (5) represents a non-trivial generalization of the conservation law for the canonical momentum of charged particle ($\vec{P}+q\vec{A}$) in an external electromagnetic field [3, 4]. I proposed in ref. [4] to name the sum $\sum_{i=1}^{N} q_i\vec{A}_i$ the "potential momentum" of a system of charged particles. At the same time, we have to be very careful in application of Eq. (4). In ref. [2] this equation has been obtained only for the particular form of vector potential of a magnetic dipole

$$\vec{A}=\frac{\vec{\mu}\times\vec{R}}{4\pi\varepsilon_0 c^2 R^3} \tag{6}$$

($\vec{\mu}$ is the magnetic moment). Eq. (4) has been also derived in ref. [4] for a non-relativistic system of mechanically free charged particles, using the Darwin gauge. However, one can see that in a general case Eq. (4) is not gauge-invariant, and hence the law of conservation of the canonical momentum has a restricted meaning. Moreover, Eq. (4) is not Lorentz-invariant. Indeed, its generalization to four-space gives

$$\frac{d}{dt}\sum_{i=1}^{N}\left(P^{\mu}\right)_{Mi}=-\frac{d}{dt}\sum_{i=1}^{N}q_i\left(A^{\mu}\right)_i \quad (\mu=0\ldots3),$$

where P^{μ} is the four-momentum of particles, and A^{μ} is the four-potential. Then we obtain the time-like component (μ=0) of this equation

$$\frac{d}{dt}\sum_{i=1}^{N}E_{Mi}=-\frac{d}{dt}\sum_{i=1}^{N}q_i\varphi_i\,, \tag{7}$$

where E_M is the mechanical energy, and φ is the electric potential. We see that the latter equation disagrees with the energy conservation law for freely moving charged particles

$$\frac{d}{dt}\sum_{i=1}^{N} E_{Mi} = -\frac{d}{dt}U ,$$

where $U = \frac{1}{2}\sum_{i=1}^{N} q_i \varphi_i$ is the potential energy of charged particles. (The multiplier ½ is absent in Eq. (7)). Nevertheless, Eq. (4) is very convenient in applications, if we assume the gauge conditions of Eq. (6) and consider only the non-relativistic case. Even so, Eq. (4) is relevant for an isolated system of *mechanically free* charged particles. If the system contains conductors and insulators with bound charges, a number of authors assumed that a so-called "hidden momentum" should be introduced in the law of conservation of total momentum (*e.g*., [5-7]). Then Eq. (4) should be transformed to

$$\sum_i \frac{d\vec{P}_{Mi}}{dt} = -\sum_i q_i \frac{d\vec{A}_i}{dt} - \frac{d\vec{Q}_h}{dt} , \tag{8}$$

where $\vec{Q}_h = \sum_{j=1}^{N_b} \vec{m}_{bj} \times \vec{E}_j \big/ c^2$ is the hidden momentum, N_b is the number of magnetic momenta with bound charges in the isolated system, and $\vec{E}_j$ is the electric field acting on the momentum $\vec{m}_{bj}$.

In this paper we analyze the problem of "hidden momentum" and arrive at the conclusion that for quasi-static configurations, which imply a balance of electromagnetic and mechanical forces, the "hidden" momentum plays an important role in the implementation of the momentum and energy conservation laws.

2. "Hidden momentum": general analysis

We again emphasize that Eq. (4) is derived for an isolated system of mechanically free charged particles [4]. If the system contains any conductors, in general, they acquire the charges of polarization with the surface charge density σ_p, which should be included in Eq. (4):

$$\frac{d}{dt}\sum_i \vec{P}_{Mi} = -\frac{d}{dt}\sum_i q_i \vec{A}_i - \frac{d}{dt}\int_S \sigma_p\left(\vec{r},t\right)\vec{A}\left(\vec{r},t\right)dS\,, \tag{9}$$

where the integration is carried out over the surface S of all conductors. In particular, when a conductor represents a magnetic dipole $\vec{\mu}$ included in a very small volume, the integral in Eq. (9) is equal to $\left(\vec{\mu}\times\vec{E}\right)/c^2$, where $\vec{E}$ is the electric field at the location of $\vec{\mu}$ [7]. The authors of ref. [7] named the value $\left(\vec{\mu}\times\vec{E}\right)/c^2$ "hidden momentum" of a conducting magnetic dipole. In my opinion, it is a matter of terminology solely, and we can always directly apply Eq. (9) for polarized conductors to get correct physical results without any reference to "hidden momentum".

An actual problem emerges when the system includes bound charges fixed on insulators. For such a case Shockley and James invented a paradox as follows [5].

Two counter-rotating oppositely charged insulating disks, whose rotation is slowed down by mutual friction, are in the electric field of a charged particle, which rests in the laboratory (Fig. 1). The particle and the disks lie in the plane xy. We want to compute the force acting on the charged particle, as well as the force acting on the whole isolated system "particle + rotating disks".

For the sake of simplicity we assume that the charge is homogeneously distributed over the perimeter of the disks. The rotational axis of the disks z passes through the point x, y=0, and at the initial instant the charge has coordinates $\{0, R, 0\}$. The radius of each disk is $r_0 \ll R$.

Let initially the rotational angular frequency of both disks be equal to ω, and the magnetic moment $\vec{\mu}$ be parallel to the axis z. Then ω

slowly decreases to zero, so that the EM radiation is negligible. During the time τ of frequency decrease, the vector potential of both disks also decreases with time, and induces an azimuthal electric field along the circumference R

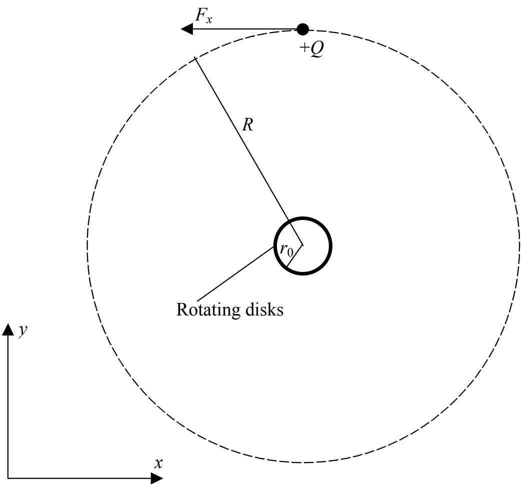

Fig. 1. The Shockley-James paradox.

$$\vec{E}(R) = -\partial\vec{A}/\partial t .$$

Taking into account Eq. (6), and the equality

$$\vec{\mu} = r_0{}^2 q\vec{\omega}$$

(q is the total charge of each disk), we derive the force, experienced by the particle Q:

$$F_x = QE_x = \frac{r_0{}^2 qQ}{4\pi\varepsilon_0 R^2}\frac{d\omega}{dt} .$$

From this the total mechanical momentum acquired by the particle Q during decrease of the rotating frequency from ω to 0 is

$$P_{Qx} = \int_0^\tau F_x dt = \frac{r_0^{\ 2} qQ}{4\pi\varepsilon_0 R^2} \int_0^\tau \frac{d\omega}{dt} dt = -QA(R). \qquad (10)$$

We see that the mechanical momentum of the charged particle after annihilation of the magnetic dipole (Eq. (10)) coincides with the potential momentum of the system -$QA(R)$ before annihilation. Shockley and James noticed that a motion of the particle Q for resting axis of the disks means a motion of the center of mass of the entire system "disks + particle", which seems to contradict special relativity. In order to resolve this paradox, they introduced a "hidden momentum" of the disks $\vec{P}_h = \left(\vec{\mu} \times \vec{E}\right)/c^2$, which exists before annihilation of $\vec{\mu}$ due to mechanical stresses in the disks. One can check that $P_{hx} = QA(R)$, and the disks acquire the same mechanical momentum P_{hx} after annihilation of $\vec{\mu}$. As a result, the center of mass of the system remains at rest.

The problem of "hidden momentum" was considered in more detail by Aharonov et al. [6] in connection with a classical model of the neutron. As a basic point, they proved a theorem as follows:

(a) There is zero total momentum (electromagnetic plus mechanical) in the rest frame of any finite static configuration, containing charged particles and magnetic momenta.

(b) There is non-vanishing electromagnetic momentum of this configuration.

In order to prove part (a) of the theorem, the authors used the requirement

$$\partial_\mu T^{\mu\nu} = \partial_\mu T_{EM}^{\ \ \mu\nu} + \partial_\mu T_M^{\ \ \mu\nu} = 0, \qquad (11)$$

where μ=0…3, $T_M{}^{\mu\nu}$ is the mechanical part of the energy momentum stress density tensor $T^{\mu\nu}$, while $T_{EM}{}^{\mu\nu}$ is the electromagnetic part satisfying (in MKSA units)

$$\partial_\mu T_{EM}{}^{\mu\nu} = -F^{\nu\lambda} j_\lambda$$

($F^{\nu\lambda}$ is the electromagnetic tensor, j_λ is the current density). The total momentum is $P^i = \frac{1}{c}\int T^{i0} dV$.

It follows from Eq. (11) that for the stationary case $\partial_i T^{i0} = 0$ (i=1…3). Then one can easily prove that

$$\vec{P} = 0 . \qquad (12)$$

On the other hand, proving part (b) of the theorem, the authors of [6] derived

$$\vec{P}_{EM} = \vec{E} \times \vec{\mu}/c . \qquad (13)$$

It follows from Eqs. (12) and (13) that

$$\vec{P}_M = \vec{\mu} \times \vec{E}/c . \qquad (14)$$

The mechanical momentum (14) represents a "hidden momentum" of the configuration, and it should be attributed to the magnetic dipole $\vec{\mu}$ exclusively. Hence, we have to use Eq. (8) instead of Eq. (4). Then one can see that Eq. (8) fully resolves the Shockley-James paradox: the disks do indeed move in the opposite direction of the charge due to their hidden momentum, and the center of mass does remain at rest.

One should notice that the manifestation of hidden momentum is model dependent [6]. In the model of a magnetic dipole involving counter-rotating charged insulating disks, the external electric field causes the mechanical stresses, as mentioned in [6]. A Lorentz transformation of the stress-energy tensor converts stress to momentum density. This contribution leads to the net mechanical momentum (14) in the rest frame of the center of the disks.

A resolution of the Shockley James paradox (a rest of the center of mass) signifies that we recover the principle of equality of action and reaction for the system "charge plus magnetic dipole". It follows from there that no net external force was applied during assembly of the configuration. However, this statement contradicts the recognized resolution of the Lewis-Tolman paradox [9], and many authors continue to reject the conception of "hidden" momentum.

In the following section we analyse two physical problems and show that the "hidden" momentum is strongly required to fulfil the energy and momentum conservation laws in EM interaction.

3. "Hidden momentum": particular physical problems

Let a charged particle q orbit around a tall solenoid S at the constant angular frequency ω (Appendix A, Fig. 3 [4]). In this problem the net force, acting on the particle, is equal to zero (except for the external force that controls the circular motion of the charge), while its "momentum" $\vec{P}_A = q\vec{A}$ changes with time. Moreover, this value defines the potential momentum of the whole system "charged particle + solenoid". Then it follows from Eq. (4) that the total time derivative $-d\vec{P}_A/dt$ should be equal to the Lorentz force, acting on the solenoid due to the particle. This result is confirmed by the particular calculations, presented in Appendix A. Thus, if no "hidden" momentum exists, a non-radiating charged particle, rotating around a solenoid, experiences no force, but induces a forced motion of the solenoid. In turn, this forced motion can perform work, which becomes infinite for infinitely long rotation of the particle. This paradox again requires introducing a hidden momentum of solenoid, where Eq. (4) is replaced by Eq. (8), and the solenoid experiences an equal reactive force. The origin of "hidden" momentum of solenoid is the mechanical stresses in its charged cylinders due to the electric field of the moving charge. During a rotation of the charge this electric field changes its spatial direction, causing a change in the direction of the mechanical stresses and the associated mechanical momentum of the solenoid.

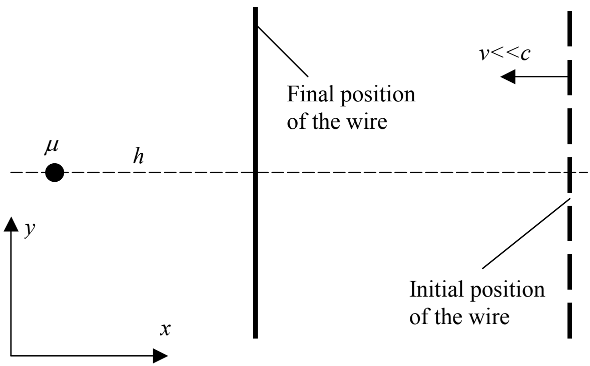

Fig. 2. Assembling of the system "magnetic dipole plus charged wire" and further annihilation of the magnetic moment $\vec{\mu}$. The magnetic dipole represents two counter-rotating oppositely charged insulating disks with a very small radius, as in Fig. 1.

Further, let us consider the problem shown in Fig. 2. An electrically neutral magnetic dipole $\vec{\mu}$ and a very long uniformly charged wire with the length L initially are separated by a large distance x_0. Nevertheless, $x_0 << L$. Initially the dipole and wire rest in the laboratory, so that there is no interaction between them. The dipole and wire lie in the *xy*-plane, the wire is parallel to the axis *y*, and the magnetic moment $\vec{\mu}$ is parallel to the axis *z*. Then the wire acquires a very small velocity *v* in the negative *x*-direction, and it is driven up to the distance *h* from the dipole, while the dipole is maintained fixed in space. After this stage of assembling, the velocity of wire becomes equal to zero, and any forces exerted on the system are vanishing. Further, the magnetic moment slowly decreases from $\vec{\mu}$ to zero during the time τ (the duration of annihilation of magnetic dipole).

The process of annihilation is similar to the Shockley-James problem, and the force, acting on the wire can be found as

$$\vec{F}_w = -\int_{-\infty}^{\infty} \lambda \frac{\partial \vec{A}(h,y,t)}{\partial t} dy,$$

where λ is the linear charge density of the wire, and $\vec{A}$ is the vector potential of magnetic dipole in the point (x=h; y; z=0) at the instant t. Then the mechanical momentum of the wire after annihilation is

$$\vec{P}_{Mw} = -\int_{t=0}^{t=\tau}\int_{-\infty}^{\infty} \lambda \frac{\partial \vec{A}(h,y,t)}{\partial t} dydt .$$

Since the vector potential of the magnetic dipole is defined by Eq. (6), we get after straightforward calculations:

$$(P_{Mw})_x = 0,\ (P_{Mw})_y = \frac{\lambda\mu}{2\pi\varepsilon_0 c^2 h} = \frac{\mu E}{c^2},$$

where $E = \lambda/2\pi\varepsilon_0 h$ is the value of electric field of the wire at the location of the magnetic dipole. This field has only an *x*-component in the plane *xy*, as we assumed that the wire is very long.

According to the conception of "hidden momentum", the magnetic dipole should acquire the mechanical momentum

$$(P_{Md})_x = (\vec{\mu}\times\vec{E})_x / c^2 = 0, (P_{Md})_y = (\vec{\mu}\times\vec{E})_y / c^2 = -\mu E/c^2 .$$

This appears due to the transformation of mechanical stresses in the decelerated disks of the magnetic dipole into a linear mechanical momentum, as in the Shockley-James problem. In such a case the center of mass of the system remains at rest. It inevitably follows from this that the total mechanical momentum, transmitted to the system during its assembling should be equal to zero. It is defined by the forces acting on the wire and dipole during the assembling process (the motion of the wire at the constant velocity v in the negative x-direction). While the wire is moving, it experiences the Lorentz magnetic force

$$\vec{F}_w = \int_{-\infty}^{\infty} \lambda\left(\vec{v}\times\vec{B}\right) dy\,, \tag{15}$$

where $\vec{B}$ is the magnetic field of magnetic dipole. In the plane xy this field has only the z-component

$$B_z = -\mu/4\pi\varepsilon_0 c^2 r^3\,. \tag{16}$$

Hence, the force (15) has a single component along the axis y:

$$\left(F_w\right)_y = -\frac{\lambda\mu v}{4\pi\varepsilon_0 c^2}\int_{-\infty}^{\infty}\frac{1}{\left(x^2+y^2\right)^{3/2}}dy = -\frac{\lambda\mu v}{2\pi\varepsilon_0 c^2 x^2}\,. \tag{17}$$

In order to prevent a motion of the wire along the axis y during the assembling of system, we have to apply to the wire a compensating counter-force

$$\left(F'_w\right)_y = \frac{\lambda\mu v}{2\pi\varepsilon_0 c^2 x^2}\,.$$

Then the mechanical momentum transmitted to the wire due to the compensating force is

$$\left(P_w\right)_y = \int_t \left(F_w'\right)_y dt = \int_t \frac{\lambda\mu v dt}{2\pi\varepsilon_0 c^2 x^2} = \int_t \frac{\lambda\mu dx}{2\pi\varepsilon_0 c^2 x^2} = \frac{\lambda\mu dx}{2\pi\varepsilon_0 c^2 h} = \frac{\mu E}{c^2}. \quad (18)$$

Next compute the Lorentz force, acting on the magnetic dipole due to the moving charged wire. Since the dipole is electrically neutral, it does not experience an electric force. In order to find the magnetic force exerted on the magnetic dipole, we assume that both charged rotating disks, constituting this dipole, are very thin and the distance between them is negligible. Hence we can adopt that both rotating disks lie in the plane *xy*. It means that in each point of both disks the electric field $\vec{E}$ of the long charged wire is parallel to the axis *x*. Since the velocity of the wire $\vec{v}$ also lies along the axis *x*, the vector product $\vec{v}\times\vec{E}=0$ at any point of the dipole. Hence, according to the field transformations (see, *e.g.* [1]), all components of the magnetic field of the moving wire are equal to zero for any point of the magnetic dipole. It follows from this that the magnetic Lorentz force experienced by the magnetic dipole is equal to zero. Hence Eq. (18) determines the total mechanical momentum transmitted to the system through the Lorenz force. However, as the wire moves, its electric field increases at the location of the dipole. Therefore, the pressure gradient (which balances the force exerted by the electric field) increases, too [10]. Hence the size of the "hidden" momentum increases appropriately. Thus, in spite of the null Lorentz force, the dipole recoils. Therefore, in order to keep the dipole in its place, a second external force, having a non-vanishing *y*-component, should be applied. This force balances the force (17) exerted on the wire. Hence the total momentum transmitted to the system vanishes.

This experiment presents an inverse effect of the Shockley-James one. In the latter effect the recoil appears when the pressure gradient and the associated "hidden" momentum decrease, whereas here it takes place when they increase.

4. Conclusion

Thus, for quasi-static macroscopic systems, which involve the electromagnetic and mechanical forces, a "hidden" momentum plays an important role, providing the implementation of the energy-momentum conservation law. In order to substantiate this assertion, we considered two particular physical problems: a charged particle rotating around a tall solenoid, and a magnetic dipole plus long charged wire. These examples demonstrate two different and counter-intuitive examples involving "hidden" momentum. In the solenoid case a Lorentz force is exerted on it but no translational acceleration takes place because the change in the momentum is absorbed in the mechanical "hidden" momentum of the rotating charges. It prevents an infinite forced motion of the solenoid. In the wire's case, no Lorentz force is exerted on the magnetic dipole, but it recoils, because the "hidden" momentum increases. In both cases, momentum conservation holds. This means that the "hidden momentum" forbids a translational motion of quasi-static configurations during a mutual transformation of electromagnetic and mechanical energies. However, as was experimentally proven in Ref. [11] and substantiated in Ref. [7], a mutual transformation of mechanical and EM energies can give rise to a rotational motion of quasi-static configurations with mutual transformation of mechanical and EM angular momenta.

Finally, for a system of mechanically free charged particles (when any static configurations are impossible), Eq. (4) holds true with the clarified above restrictions. This signifies a violation of Newton's third law in electromagnetic interaction, and hence both mechanical momentum and mechanical angular momentum can be transformed into their electromagnetic counterparts.

Acknowledgments

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Appendix A. Calculation of the Lorentz force acting on a solenoid due to a charged particle rotating around the solenoid

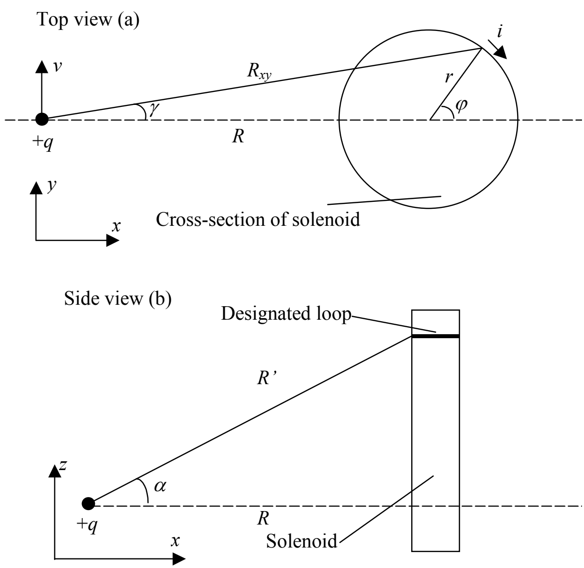

Fig. 3. The charged particle +*q* orbits around the solenoid. We imagine a solenoid as two oppositely charged insulated elongated cylinders with thin walls and equal radius, which rotate without friction at the opposite directions about a common axis and the angular frequency ω. The charges are rigidly fixed on the insulating walls, which allows excluding the charge of polarization. In order to apply the same expressions, as for a conducting solenoid, we assume that each cylinder contains *N* equally charged layers with the charge *Q* and with *n* layers per unit length. Then the current in each layer is equal to $i=Q\omega/2\pi$.

Let a charged particle q orbit in the xy-plane around a tall solenoid S at the constant angular frequency ω (Fig. 3). The radius of the solenoid is equal to r, the distance between the particle and axis of solenoid is $R>r$. The axis of the solenoid is parallel to the axis z. One requires determining the force experienced by the solenoid due to the charged particle.

In the non-relativistic limit a moving charged particle creates the magnetic field

$$\vec{B} = \frac{q\vec{v} \times \hat{\vec{n}}}{4\pi\varepsilon_0 c^2 R'^2}, \quad \text{(A1)}$$

where $\hat{\vec{n}}$ is the unit vector, joining the point-like charge and designated space point R'. This field induces a magnetic force, acting on each element dl of solenoid with the current $\vec{i}$:

$$d\vec{F} = \left(\vec{i} \times \vec{B}\right) dl \, . \quad \text{(A2)}$$

Without loss of generality, we can choose the axis x to be orthogonal to $\vec{v}$ at t=0. Then Eq. (A1) gives the following components of magnetic field:

$$B_x = \frac{qvn_z}{4\pi\varepsilon_0 c^2 R'^2}, \; B_y = 0, \; B_z = -\frac{qvn_x}{4\pi\varepsilon_0 c^2 R'^2}. \quad \text{(A3)}$$

Also taking into account, that i_z=0 in the solenoid, we obtain the components of force for a single loop of the solenoid as

$$dF_{lx} = i_y B_z dl = -iB_z r \cos\varphi d\varphi \, , \quad \text{(A4)}$$

$$dF_{ly} = -i_x B_z dl = -iB_z r \sin\varphi d\varphi \, , \quad \text{(A5)}$$

$$dF_{lz} = -i_y B_x dl = iB_x r \cos\varphi d\varphi \, . \quad \text{(A6)}$$

where φ is the azimuthal angle (Fig. 3, a). Firstly, let us calculate the component of total force along the axis x. Substituting B_z from Eqs. (A3) into Eq. (A4), we obtain

$$dF_{lx} = \frac{qvn_x ir\cos\varphi d\varphi}{4\pi\varepsilon_0 c^2 R'^2}. \tag{A7}$$

One can see from Fig. 3, that

$$R'^2 = R_{xy}{}^2 + z^2,\ R_{xy}{}^2 = R^2 + 2Rr\cos\varphi + r^2,\ \cos\alpha = \frac{R_{xy}}{R'},$$

$$\cos\gamma = \frac{R + r\cos\varphi}{R_{xy}}, n_x = \cos\alpha\cos\gamma = \frac{R + r\cos\varphi}{\sqrt{R^2 + 2Rr\cos\varphi + r^2 + z^2}}. \tag{A8}$$

Substituting the values of (A8) into (A7), one gets:

$$dF_x = \frac{qvir\left(R + r\cos\varphi\right)\cos\varphi d\varphi}{4\pi\varepsilon_0 c^2\left(R^2 + 2Rr\cos\varphi + r^2 + z^2\right)^{3/2}}. \tag{A9}$$

Then we derive for a single loop

$$dF_{lx} = \frac{qvir}{4\pi\varepsilon_0 c^2}\int_0^{2\pi}\frac{\left(R + r\cos\varphi\right)\cos\varphi d\varphi}{\left(R^2 + 2Rr\cos\varphi + r^2 + z^2\right)^{3/2}}$$

$$= \frac{qvir}{4\pi\varepsilon_0 c^2 R}\int_0^{2\pi}\frac{\left(1 + \frac{r}{R}\cos\varphi\right)\cos\varphi d\varphi}{\left(1 + \frac{2r}{R}\cos\varphi + \frac{r^2}{R^2} + \frac{z^2}{R^2}\right)^{3/2}}.$$

The fragment of solenoid with the length dz contains ndz layers (loops). Hence, the force, acting on the fragment with the length dz is

$$dF_x = \frac{qvirndz}{4\pi\varepsilon_0 c^2 R} \int_0^{2\pi} \frac{\left(1 + \frac{r}{R}\cos\varphi\right)\cos\varphi d\varphi}{\left(1 + \frac{2r}{R}\cos\varphi + \frac{r^2}{R^2} + \frac{z^2}{R^2}\right)^{3/2}}.$$

From this the total force acting on the solenoid along the axis x is

$$F_x = \frac{qvirn}{4\pi\varepsilon_0 c^2 R} \int_{-\infty}^{\infty} \int_0^{2\pi} \frac{\left(1 + \frac{r}{R}\cos\varphi\right)\cos\varphi d\varphi dz}{\left(1 + \frac{2r}{R}\cos\varphi + \frac{r^2}{R^2} + \frac{z^2}{R^2}\right)^{3/2}}.$$

Taking into account that $\frac{in}{\varepsilon_0 c^2} = B$, we obtain

$$F_x = \frac{qvrB}{4\pi R} \int_{-\infty}^{\infty} \int_0^{2\pi} \frac{\left(1 + \frac{r}{R}\cos\varphi\right)\cos\varphi d\varphi dz}{\left(1 + \frac{2r}{R}\cos\varphi + \frac{r^2}{R^2} + \frac{z^2}{R^2}\right)^{3/2}}.$$

This equation can also be expressed via the value of vector potential of solenoid A, using the equality $A = Br^2/R$ (outside the solenoid):

$$F_x = \frac{qvA}{2\pi r} \int_{-\infty}^{\infty} \int_0^{2\pi} \frac{\left(1 + \frac{r}{R}\cos\varphi\right)\cos\varphi d\varphi dz}{\left(1 + \frac{2r}{R}\cos\varphi + \frac{r^2}{R^2} + \frac{z^2}{R^2}\right)^{3/2}}. \qquad \text{(A10)}$$

Integration over z gives:

$$F_x = \frac{qvAR^2}{2\pi r} \int_0^{2\pi} \left(1 + \frac{r}{R}\cos\varphi\right)\cos\varphi d\varphi \int_{-\infty}^{\infty} \frac{dz}{\left(R^2 + 2rR\cos\varphi + r^2 + z^2\right)^{3/2}}$$

$$= \frac{qvAR^2}{\pi r} \int_0^{2\pi} \frac{\left(1 + \frac{r}{R}\cos\varphi\right)}{R^2 + 2rR\cos\varphi + r^2}\cos\varphi d\varphi \qquad \text{(A11)}$$

$$= \frac{qvA}{\pi r} \int_0^{2\pi} \frac{\left(1 + \frac{r}{R}\cos\varphi\right)}{1 + \frac{2r}{R}\cos\varphi + \frac{r^2}{R^2}}\cos\varphi d\varphi$$

The remaining integral over φ is equal to

$$\int_0^{2\pi} \frac{\left(1 + \frac{r}{R}\cos\varphi\right)}{1 + \frac{2r}{R}\cos\varphi + \frac{r^2}{R^2}}\cos\varphi d\varphi = -\frac{\pi r}{R}. \qquad \text{(A12)}$$

Substituting Eq. (A12) into Eq. (A11), we obtain

$$F_x = -qvA/R. \qquad \text{(A13)}$$

Similar calculations show that the *y*- and *z*-components of force, defined by Eqs. (A5) and (A6), correspondingly, are both equal to zero. Hence, Eq. (A13) describes the total momentary Lorentz force due to the rotating particle with a negative *x*-coordinate, when the axis *x* is orthogonal to its orbital velocity. It shows that the force is directed along the line joining the axis of the solenoid and the momentary position of the rotating particle. It follows from there that the direction of the force, exerted by the particle on the solenoid, rotates together with the particle at the same angular frequency ω. Hence, the projections of this force change with time for a laboratory observer as

$$F_x = qvA/R = q\omega A\cos\omega t \qquad \text{(A14)}$$

$$F_y = qvA/R = q\omega A \sin \omega t \tag{A15}$$

and $F_z = 0$.

One can see that Eqs. (A14) and (A15), taken together, can be written in the vector form as

$$\vec{F} = -q\left(\vec{\omega} \times \vec{A}\right). \tag{A16}$$

For the vector field of the solenoid we write

$$\frac{d\vec{A}}{dt} = \left(\vec{\omega} \times \vec{A}\right). \tag{A17}$$

Comparison of Eqs. (A16) and (A17) yields

$$\vec{F} = -q\, d\vec{A}/dt = -d\vec{P}_A/dt\,. \tag{A18}$$

Thus, we have shown that the Lorentz force, acting on the solenoid due to a rotating particle, is equal with the opposite sign to the total time derivative of the potential momentum $q\vec{A}$ for the system "solenoid +particle".

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